Module 2: Numerical methods

Week 8 Tutorial

Solving partial differential equations

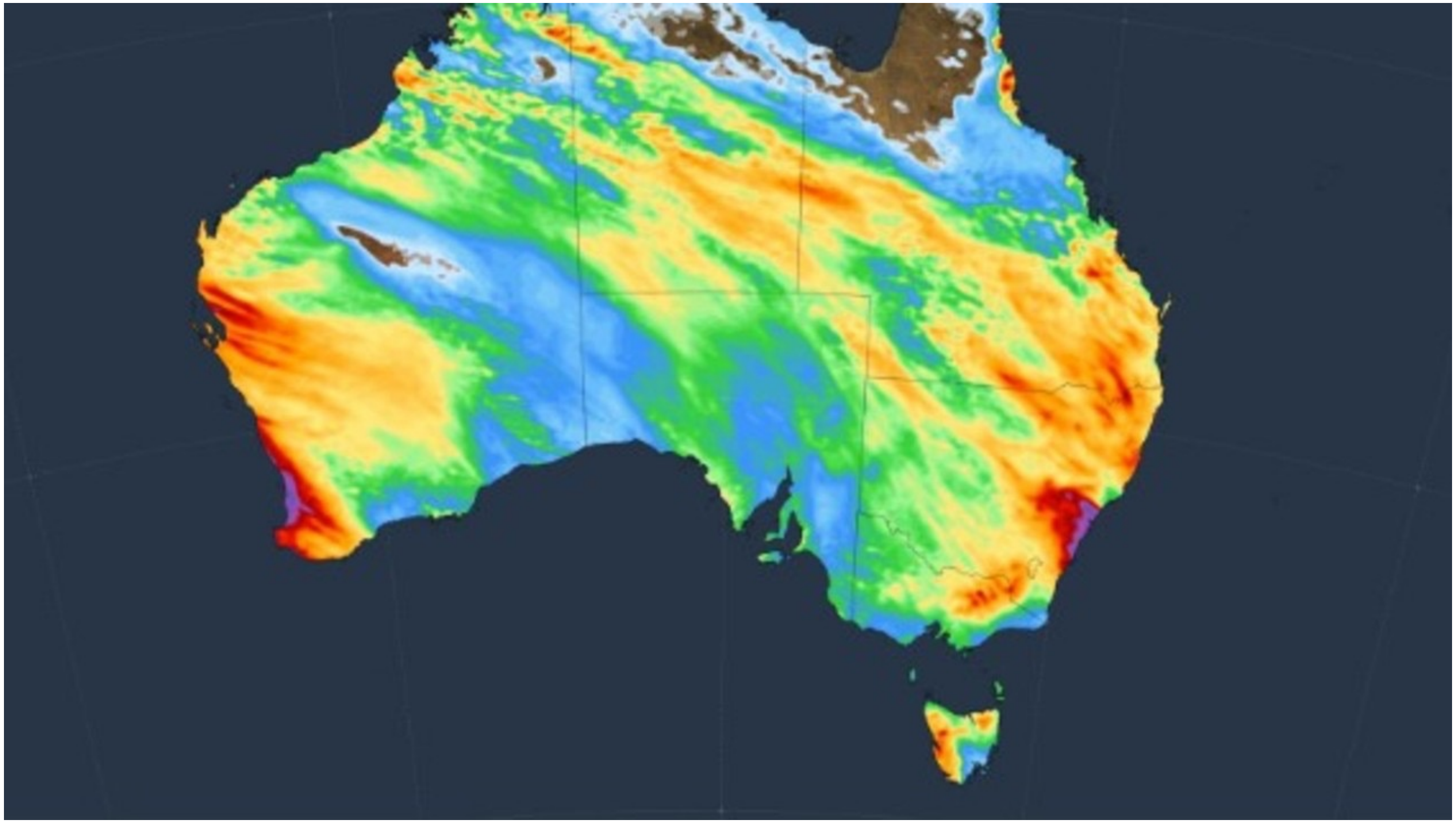
Key goals for the class

1. How do we apply finite difference methods to partial differential equations where there is **more than one independent variable**?

2. *Only one goal this week, since this one is a bit tricky !!*

Partial differential equations

Partial differential equations involve dependent variables which are functions of 2 or more independent variables:



Partial differential equations

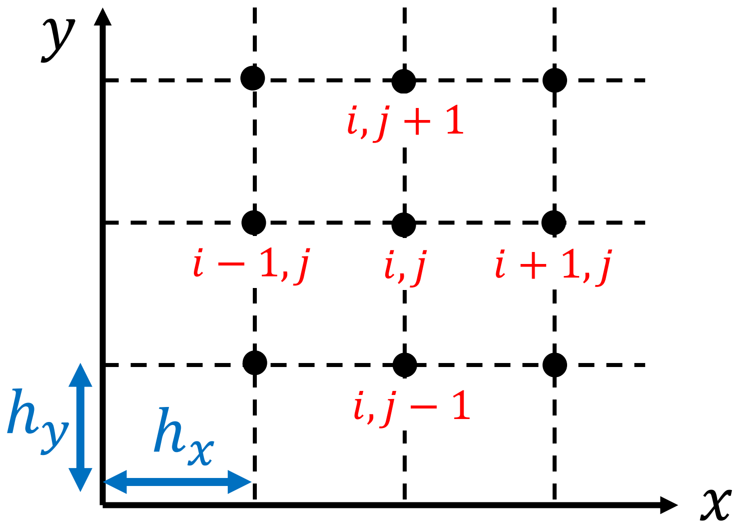
Partial differential equations involve dependent variables which are functions of 2 or more independent variables, such as $u(x, y)$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

We can numerically solve such equations by substituting the finite difference expressions on a **2D grid** $u_{i,j}$ *(now 2 subscripts!)*

$$\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2}$$

$$\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}$$

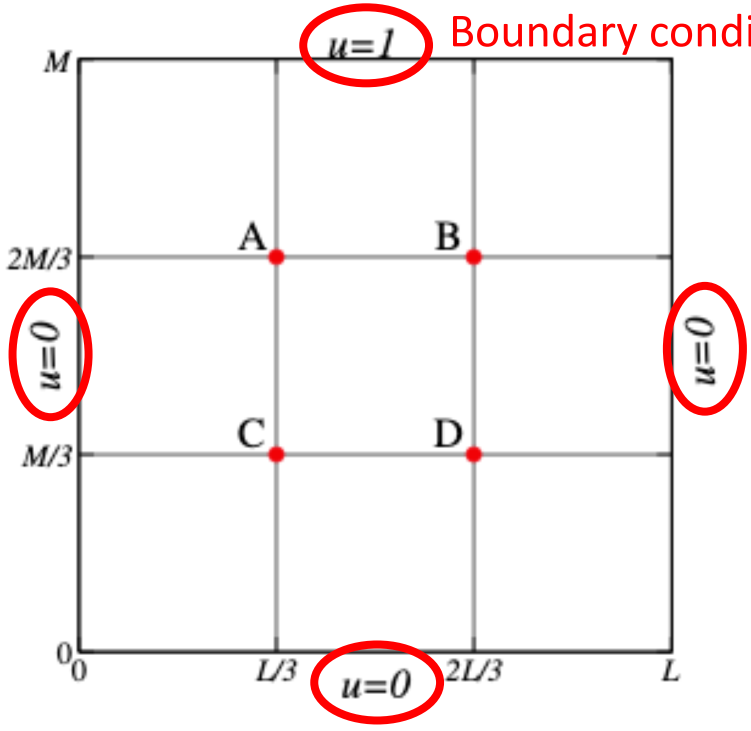


Partial differential equations

Q1 from the tutorial sheet:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

Boundary conditions!



Find u_A, u_B, u_C, u_D

Finite differences at point A...

$$\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_B - 2u_A + 0}{(L/3)^2}$$

$$\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{1 - 2u_A + u_C}{(M/3)^2}$$

Apply to the rest of the points, to make 4 equations for the 4 unknowns u_A, u_B, u_C, u_D !

We can also use a symmetry argument to say $u_A = u_B$ and $u_C = u_D$ – why does this work?

Tutorial question

Try Q1 on the tutorial sheet (**solving a PDE on a grid**).

- 1. Consider Poisson's equation $\Delta u = \nabla^2 u = -1$, with $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2$, in the $L \times M$ rectangular domain shown in Figure 1

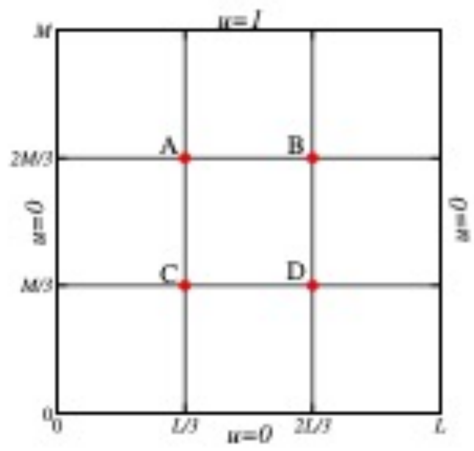


Figure 1: Domain with the boundary conditions in Q1.

with boundary conditions $u(0, y) = u(L, y) = u(x, 0) = 0$ and $u(x, M) = 1$.

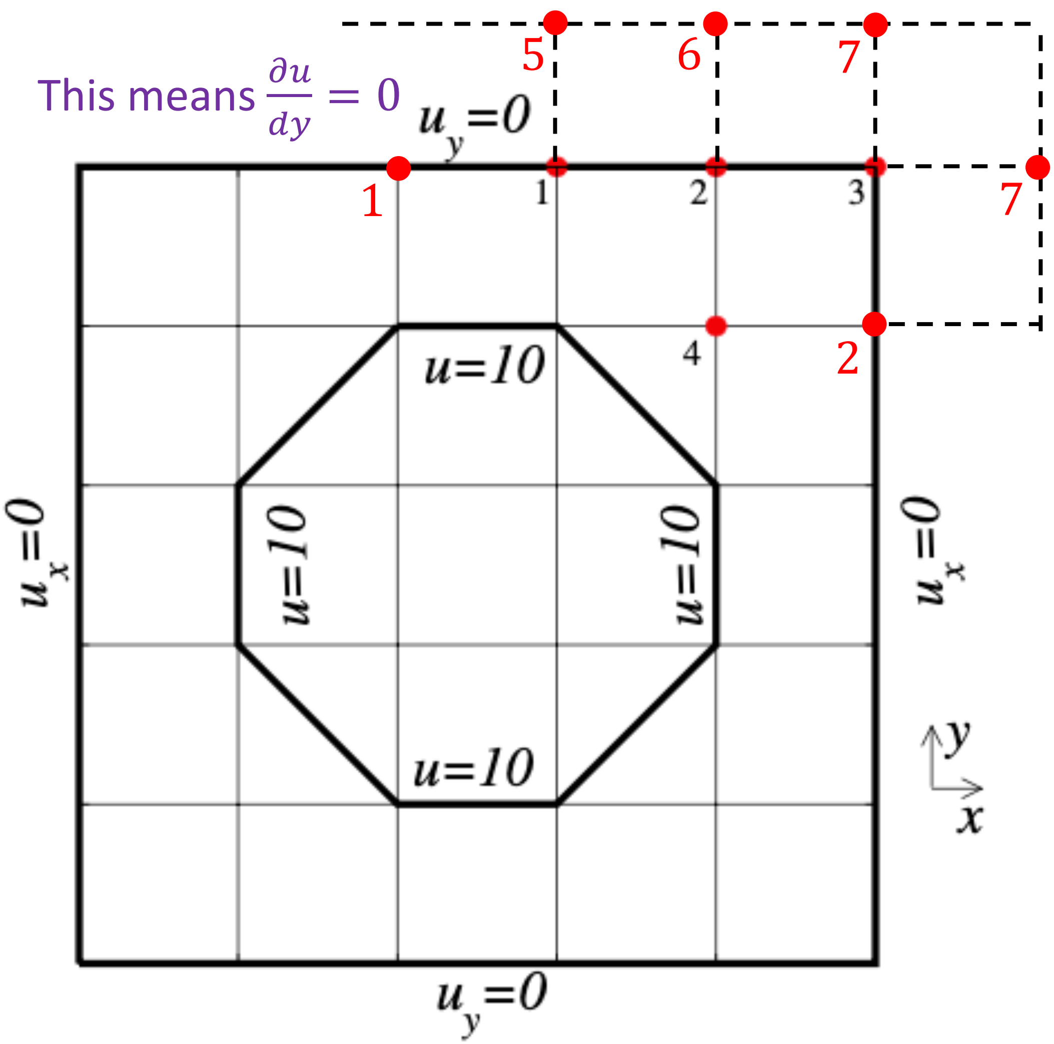
(a) Assume $M = L$ and determine the values of u at points A, B, C and D.

(b) Generalize your answer in part (a) for $M \neq L$.

Adding extra grid points

Q2 from the tutorial sheet:

This means $\frac{\partial u}{dy} = 0$



To apply finite differences, we make a “cross” around each point $\{1,2,3,4\}$, which necessitates **adding extra grid points**!

We can use **symmetry arguments** to fix the values of some of these extra points, as shown

1) Write the finite differences for $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at points $\{1,2,3,4\}$

2) Write the boundary condition $\frac{\partial u}{\partial y} = 0$ as a central difference at points $\{1,2,3\}$

3) Solve the resulting equations! (not too bad in this case!)

Tutorial question

Try Q2 on the tutorial sheet (**solving a PDE on a grid**).

- 2. Consider Laplace's equation $\Delta u = \nabla^2 u = 0$, with $\Delta = \nabla^2 = \partial_x^2 + \partial_y^2$, in the domain, shown in Figure 2. The values of the function is $u = 10$ along the inner walls of the domain. Along the vertical outer wall $u_x = \frac{\partial u}{\partial x} = 0$ and along the horizontal outer wall $u_y = \frac{\partial u}{\partial y} = 0$. Determine the values of u at points 1, 2, 3 and 4.

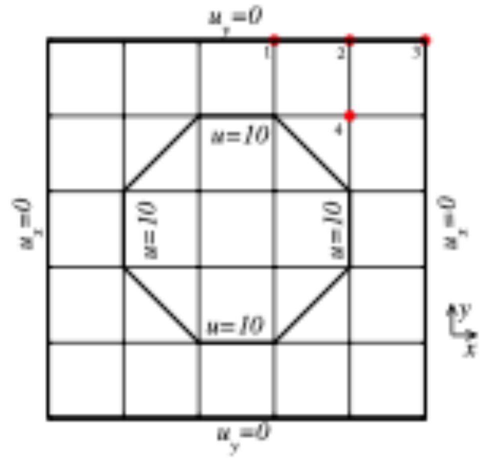


Figure 2: Domain with the boundary conditions in Q2.

Tutorial question

Try Q3 on the tutorial sheet (**if time**).

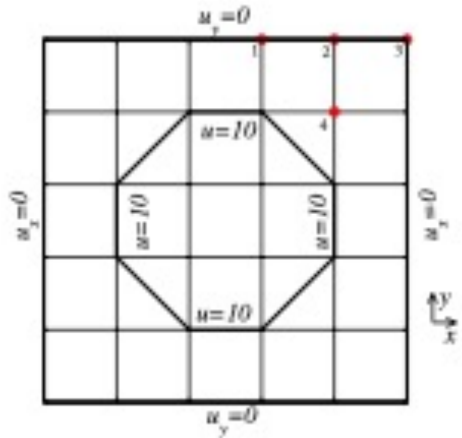


Figure 2: Domain with the boundary conditions in Q2.

- 3. (*) Stationary temperature field $u(x, y)$ in the domain in Figure 2 satisfies the Laplace equation $\Delta u = \nabla^2 u = 0$. Assume that the value of u is fixed to $u = 0$ along all inner and outer boundaries. Can you predict the value of the temperature in all points of the domain without carrying out calculations? Justify your answer.

That’s all for today!