Module 2: Numerical methods

Week 7 Tutorial

Numerical stability and boundary value problems

Key goals for the class

1. How do we determine the step size that allows a finite difference method to be **stable**?

2. How do we solve differential equations with **boundary values** using finite difference and matrix methods?

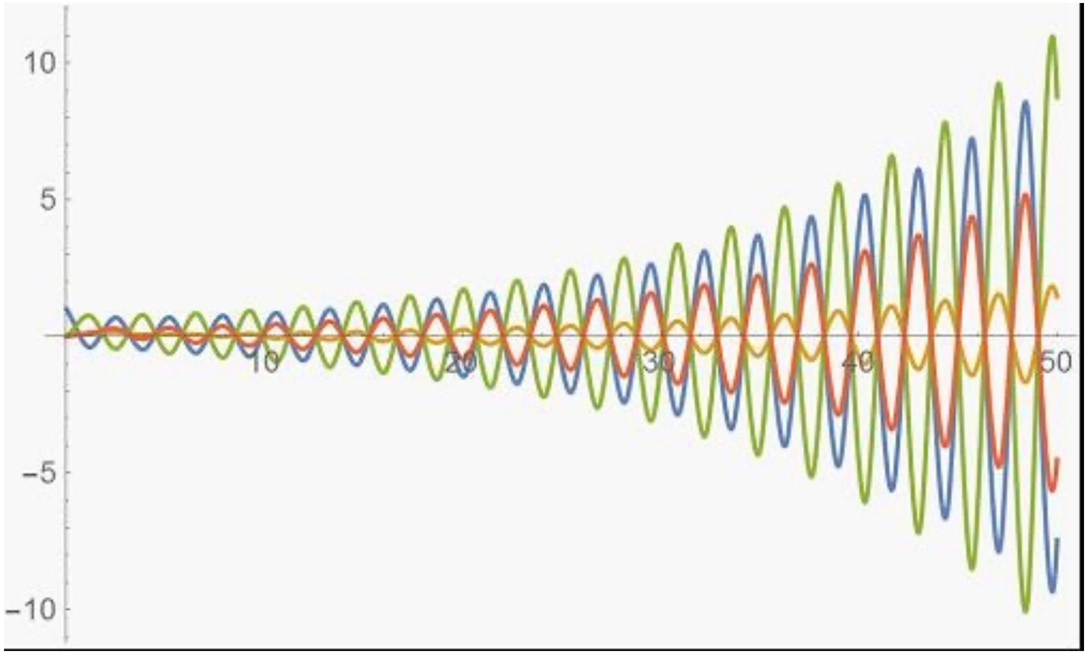
Stability of solutions

What is meant by an “unstable” finite difference solution?

What could cause this to happen?

Stability of solutions

In an unstable finite difference approximation, **numerical errors compound** such that the solution becomes exponentially less accurate



Commonly, instability results if the **step size is too high**

Stability of solutions

We can test the **stability** of a numerical solution by a qualitative comparison with an exact decaying solution:

$$\frac{df}{dx} = \lambda f \quad \rightarrow \quad f \propto e^{\lambda x} \quad \rightarrow \quad \text{decays if } \operatorname{Re}\{\lambda\} < 0$$

Express $\frac{df}{dx} = \lambda f$ using finite differences with step size h, and ask **for what values of h will the numerical solution decay**?

Method	Equation	Stability condition
Forward Euler	$\frac{f_{i+1} - f_i}{h} = \lambda f_i$	$\lvert 1 + \lambda h \rvert < 1$
Backward Euler	$\frac{f_{i+1} - f_i}{h} = \lambda f_{i+1}$	$\lvert 1 - \lambda h \rvert > 1$
Modified Euler	$\frac{f_{i+1} - f_i}{h} = \frac{\lambda}{2}(f_i + f_{i+1})$	$\operatorname{Re}\{\lambda h\} < 0$

Tutorial question

Try Q1 on the tutorial sheet (**stability of solutions**).

- 1. (a) Determine for which values of b the forward Euler method with step $h = 0.1$ is stable

$$\dot{y}(t) = (-1 + bj)\,y(t), j = \sqrt{-1}.$$

(b) Determine for which values of the discretization step h the forward Euler method is stable

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) Determine for which values of the discretization step h the forward Euler method is stable

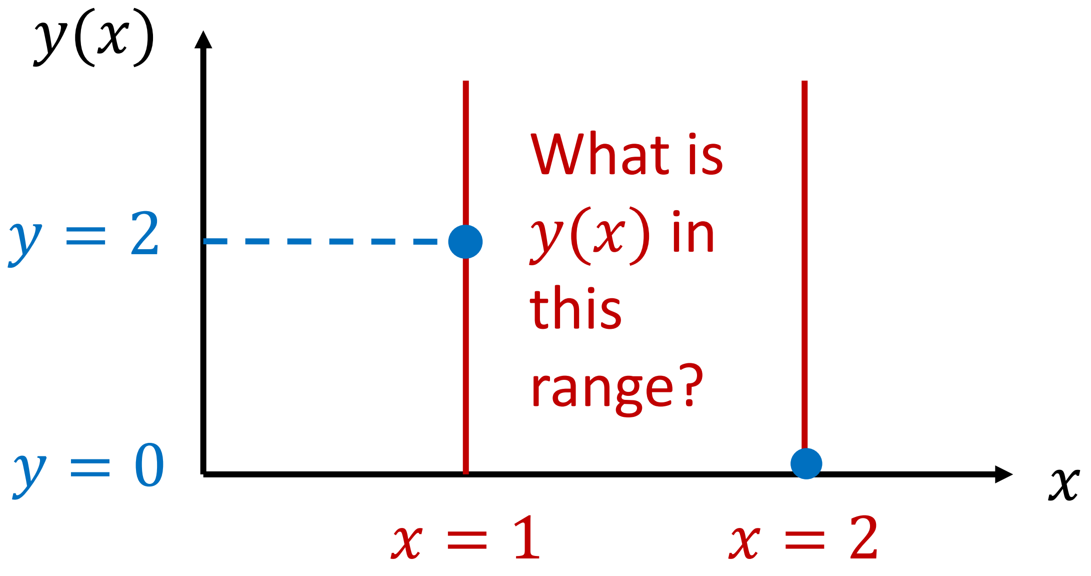
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Hint for (b) and (c): The matrix equations are eigenvalue problems because if $\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$, then substituting $\vec{x} = \vec{c}e^{\lambda t}$ leads to $\mathbf{A}\vec{c} = \lambda\vec{c}$ – the stability condition needs to apply separately for every eigenvalue

Boundary value problems

Sometimes we need to solve differential equations where conditions are specified **at the boundaries**.

e.g. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = x^2$ where $\boldsymbol{y(1) = 2}$ and $\boldsymbol{y(2) = 0}$



Boundary value problems

If we create a solution on a mesh, this naturally leads to a matrix formulation.

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = x^2$ where $\boldsymbol{y(1) = 2}$ and $\boldsymbol{y(2) = 0}$

Method:

- Replace the derivatives with finite differences
- Group the terms in y_{i-1}, y_i, y_{i+1}
- Write the resulting equation as a matrix

$$\rightarrow \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\frac{y_{i+1} - y_{i-1}}{2h}\right) - y_i = x_i^2$$

$$\rightarrow \left(\frac{1}{h^2} + \frac{1}{2h}\right) y_{i+1} - \left(\frac{2}{h^2} + 1\right) y_i + \left(\frac{1}{h^2} - \frac{1}{2h}\right) y_{i-1} = x_i^2$$

Tutorial question

Try Q2 on the tutorial sheet (**boundary value example**).

- 2. Consider the following boundary value problem

$$y''(x) + y'(x) - y(x) = x^2, \;\; y(1) = 2, \;\; y(2) = 0.$$

Use central differences with the grid spacing $h = 1/4$ to set up a system of three linear equations for the computation of the approximate values for $y(1.25)$, $y(1.5)$ and $y(1.75)$. Give the answer in the form

$$A \begin{bmatrix} y(1.25) \\ y(1.5) \\ y(1.75) \end{bmatrix} = B,$$

where A is a 3×3 matrix and B is a 3×1 column vector. DO NOT attempt to solve the system.

Tutorial question

Try Q3 on the tutorial sheet (**if time**).

- 3. Consider the following boundary value problem

$$y''(x) + y'(x) - xy(x) + 1 = 0, \quad y(0) = 0, \quad y'(1) = 1.$$

Use the central difference approximation with step $h = 1/2$ to compute $y(0.5)$ and $y(1)$.

That’s all for today!