Module 2: Numerical methods

Week 6 Tutorial

Differential equations on a finite grid

Key goals for the class

1. How do we express a **differential equation** on a finite grid?

2. How do we use finite difference methods to solve these differential equations given **initial values**?

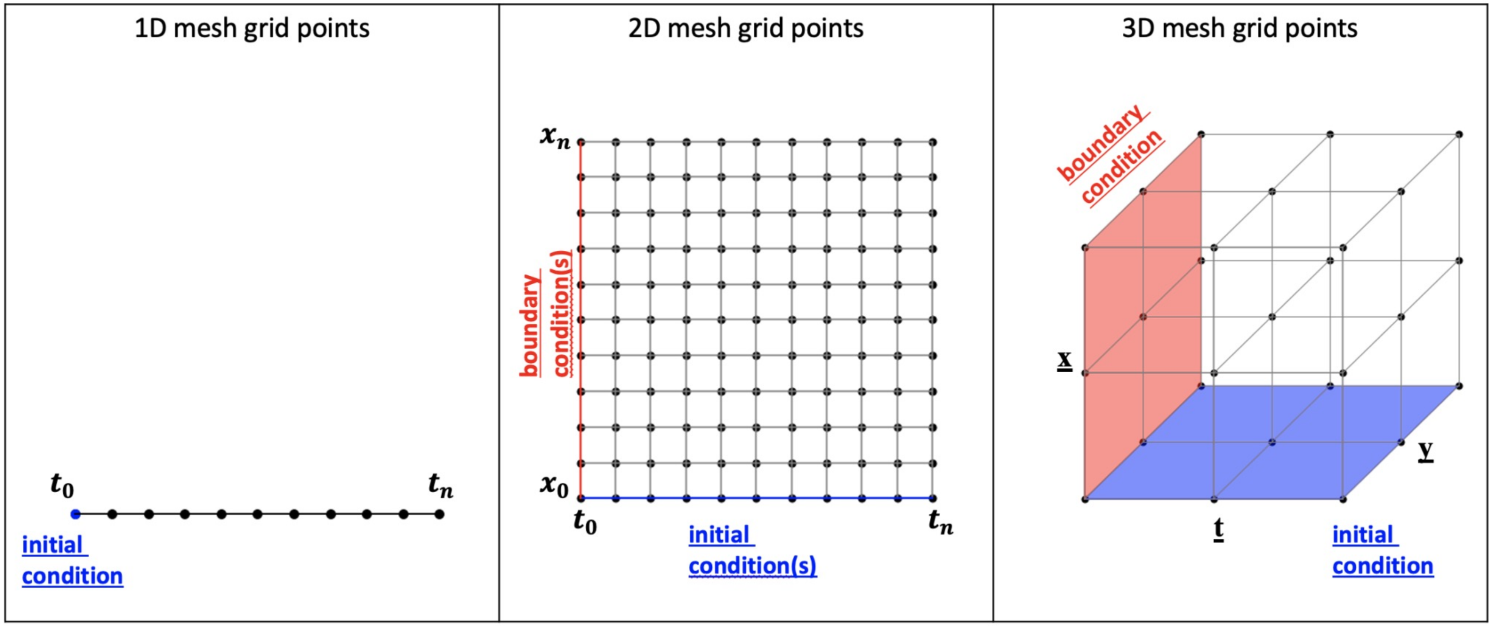
Differential equations on a grid

Differential equations

Fundamental equations describing variables in physics, maths or engineering

Discrete mesh

Grid needed to solve these problems on a computer



Differential equations on a grid

An **initial value problem** is a differential equation for $y(x)$ with initial conditions given for y, such as

$$\frac{dy}{dx} = 1 + xy, \qquad y(2) = 0$$

We want to approximate y at nearby values to $x = 2$ using a **discrete mesh**. Our method:

1. Replace the derivative with a finite difference approximation in terms of the function values at mesh points i and $i + 1$
2. Re-arrange the equation to get y_{i+1} in terms of y_i, x_i and x_{i+1}
3. Evaluate recursively for each successive mesh point

Differential equations on a grid

We can use **finite differences** to replace

$$\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$$

We can apply this derivative at point x_i, point x_{i+1} or halfway in between, $\frac{1}{2}(x_i + x_{i+1})$, creating these methods:

Forward Euler Method: $y \rightarrow y_i$

Backward Euler Method: $y \rightarrow y_{i+1}$

Modified/Central Euler Method: $y \rightarrow \frac{1}{2}(y_i + y_{i+1})$

Differential equations on a grid

Let's see those methods applied to our initial problem:

$$\frac{dy}{dx} = 1 + xy, \qquad y(2) = 0$$

Method	Application point	Resulting equation
Forward Euler Method	i	$\frac{y_{i+1} - y_i}{h} = 1 + x_i y_i$
Backward Euler Method	$i + 1$	$\frac{y_{i+1} - y_i}{h} = 1 + x_{i+1} y_{i+1}$
Modified Euler Method	Mid-point	$\frac{y_{i+1} - y_i}{h} = 1 + \frac{1}{2}(x_i y_i + x_{i+1} y_{i+1})$

Then re-arrange to get y_{i+1} in terms of y_i !

Tutorial question

Try Q1(a) on the tutorial sheet (**forward Euler method**).

- 1. Consider the initial value problem

$$\frac{dy}{dx} = 1 + xy, \quad y(2) = 0.$$

(a) Calculate the value for $y(2.2)$ using the forward Euler method with two different values of the step $h = 0.2$ and $h = 0.1$. Give the answer correct to three decimal places.

On left-hand side: $\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$

On right-hand side: $xy \rightarrow x_i y_i$

Then re-arrange to find y_{i+1} in terms of the others

Tutorial question

Try Q1(b) on the tutorial sheet (**backward Euler method**).

- 1. Consider the initial value problem

$$\frac{dy}{dx} = 1 + xy, \quad y(2) = 0.$$

(b) Calculate the value for $y(2.2)$ using the backward Euler method with $h = 0.2$. Give the answer correct to three decimal places. The backward Euler's method for $y' = f(x, y(x))$ is given by

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}), \quad x_{n+1} = x_n + h.$$

On left-hand side: $\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$

On right-hand side: $xy \rightarrow x_{i+1}y_{i+1}$

Then re-arrange to find y_{i+1} in terms of the others

Tutorial question

Try Q1(c) on the tutorial sheet (**modified Euler method**).

• 1. Consider the initial value problem

$$\frac{dy}{dx} = 1 + xy, \quad y(2) = 0.$$

(c) Calculate the value for $y(2.2)$ using the modified Euler method with $h = 0.2$. Give the answer correct to three decimal places. The modified Euler method for $y' = f(x, y(x))$ is given by

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_{n+1}, y_{n+1}) + f(x_n, y_n)\right), \quad x_{n+1} = x_n + h.$$

On left-hand side: $\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$

On right-hand side: $xy \rightarrow \frac{1}{2}(x_i y_i + x_{i+1} y_{i+1})$

Then re-arrange to find y_{i+1} in terms of the others

Tutorial question

Try Q2 on the tutorial sheet (**backward Euler method**).

- 2. The initial value problem (IVP)

$$y' = f(x, y) = (1 + x^2)y^2, \quad y(0) = -1$$

is solved using the backward Euler method with step $h = 0.1$. Take one step of the method to determine the approximate value of $y(0.1)$. Round correct to three decimal places.

On left-hand side: $\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$

On right-hand side: $x \rightarrow x_{i+1}$, $y \rightarrow y_{i+1}$

Then re-arrange to find y_{i+1} in terms of the others

Tutorial question

Try Q3 on the tutorial sheet (**forward Euler method**).

- 3. The initial value problem (IVP)

$$y' = f(x, y) = -y, \quad y(a) = b$$

is solved using the forward Euler method. Derive the formulae for y_n for arbitrary n and arbitrary step h. What can you say about $\lim_{n\to\infty} y_n$? Determine the maximal possible value of h such that $\lim_{n\to\infty} y_n = 0$.

On left-hand side: $\frac{dy}{dx} \rightarrow \frac{y_{i+1} - y_i}{h}$

On right-hand side: $y \rightarrow y_i$

Then re-arrange to find y_{i+1} in terms of the others

Tutorial question

Try Q1(d) on the tutorial sheet (**if time**).

- 1. Consider the initial value problem

$$\frac{dy}{dx} = 1 + xy, \quad y(2) = 0.$$

(d) Calculate the value for $y(2.2)$ correct to three decimal places, using the fourth-order Runge-Kutta method with $h = 0.2$ The fourth-order Runge-Kutta scheme is given by

$$\begin{aligned}
k_1 &= f(x_n, y_n), \\
k_2 &= f\left(x_n + \frac{h}{2}, y_n + k_1\frac{h}{2}\right), \\
k_3 &= f\left(x_n + \frac{h}{2}, y_n + k_2\frac{h}{2}\right), \\
k_4 &= f\left(x_n + h, y_n + h\,k_3\right), \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\
x_{n+1} &= x_n + h, \quad (n = 0, 1, 2, \dots)
\end{aligned}$$

That’s all for today!