Module 2: Numerical methods

Week 5 Tutorial

Finite difference methods

In-class test coming up on Thursday!

Test is on Week 1-4 content (matrices and eigenvalues):

- Find determinant, trace, eigenvalues, eigenvectors of a matrix
- Express determinant/trace in terms of eigenvalues
- Normalise eigenvectors and test for orthogonality
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- Express a higher-order differential equation in matrix form

Key goals for today’s class

1. How do we extrapolate a function using a **Taylor series expansion**?

2. What is the difference between a **continuous function** and a **discrete function**?

3. How do we **approximate the derivatives** of a discrete function?

Approximating a function

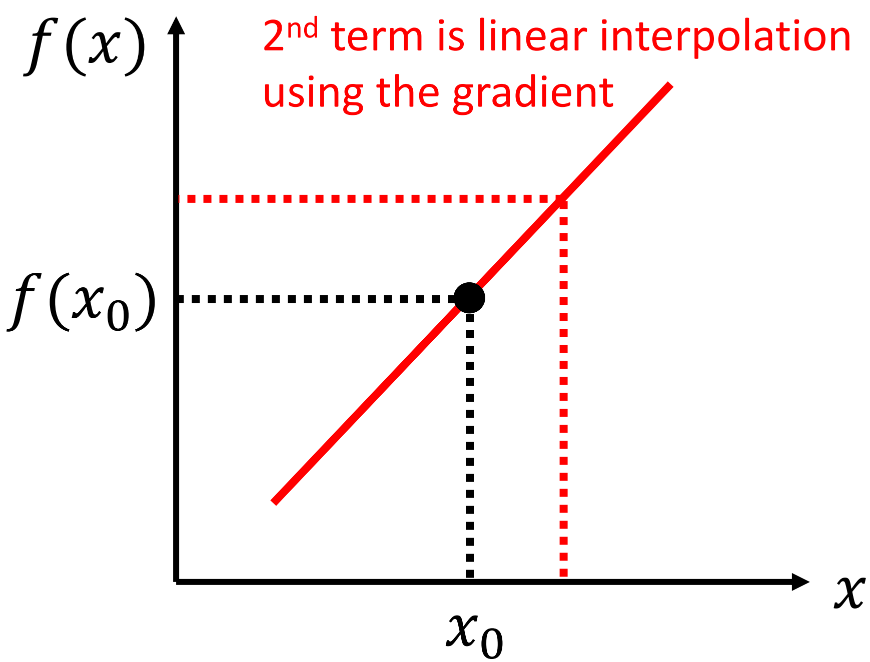
What does a "Taylor series" (sometimes called "Maclaurin series") look like?

Why is it useful?

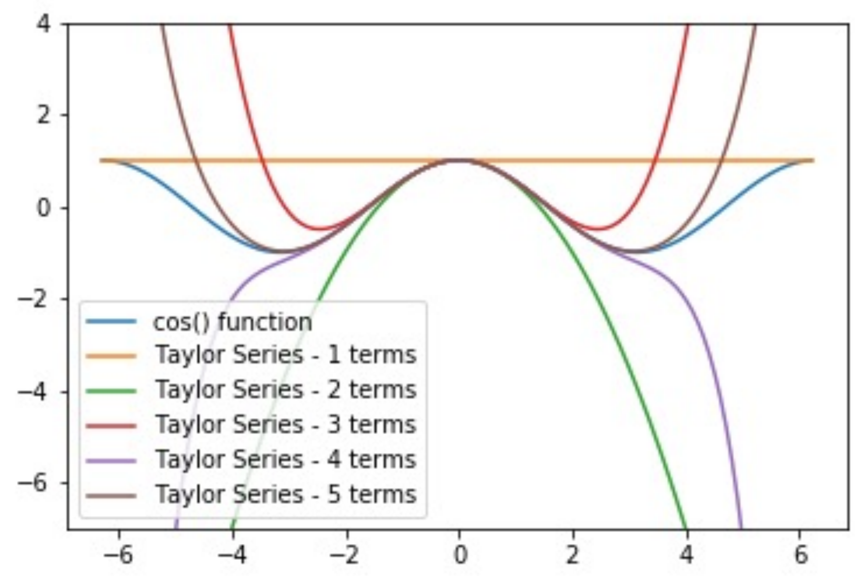
Approximating a function

Taylor series: A function may be extrapolated from a point x_0 to other x's if we know its derivatives at x_0 :

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + f''(x_0)\,\frac{1}{2}(x - x_0)^2 + \cdots$$



Remaining terms increase the accuracy …



Tutorial question

Try Q2 on the tutorial sheet (**Taylor series**).

- 2. Given $f(1) = 1$, $f(0.9) = 0.9$ and $f(1.2) = 0.9$.
  - (a) Write down the first three terms of the Taylor series for $f(x)$ about $x = 1$.
  - (b) Use the Taylor series from part (a) to express $f(0.9)$ and $f(1.2)$.
  - (c) Determine $f'(1)$ and $f''(1)$.

Hint: $f(x) = f(x_0) + f'(x_0)\,(x - x_0) + f''(x_0)\,\frac{1}{2}(x - x_0)^2$

Approximating a function

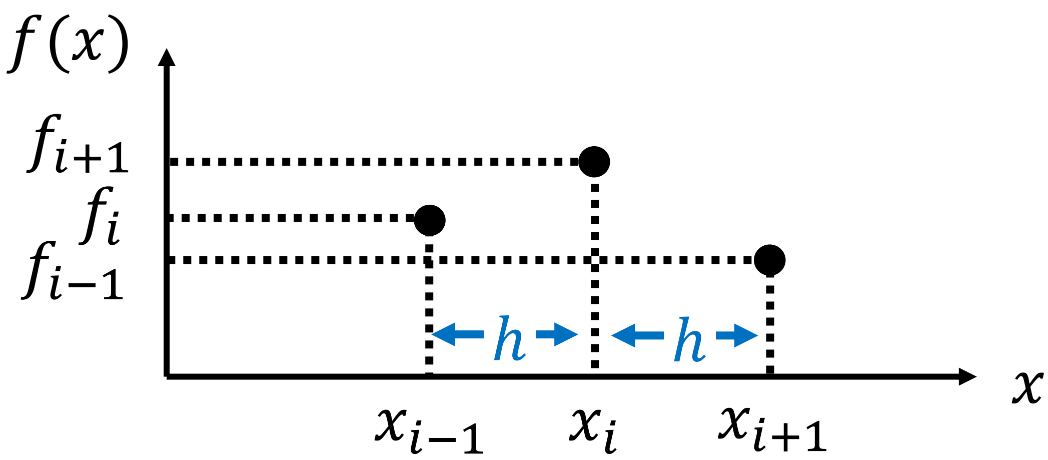
What is the difference between a **continuous function** and a **discrete function**?

Why do we need to use discrete functions for anything?

Function on a discrete mesh

A **continuous function** is defined at every point x (that is, an infinite number of points)

For a computer this is not practical, so functions are defined on a **discrete mesh** of grid points labelled $i = 1,2, \dots, N$



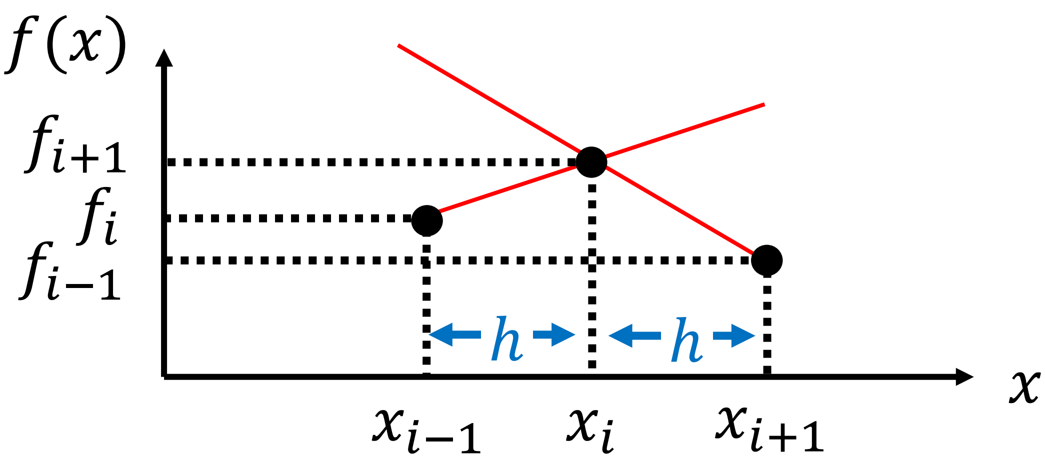
Approximating the derivatives

Derivatives of a function may be approximated from the discrete values $(i-1, i, i+1)$ spaced by h:

1st derivative, 1-sided: $f_i' = \frac{f_{i+1} - f_i}{h} + O(h)$ $\quad f_i' = \frac{f_i - f_{i-1}}{h} + O(h)$

1st derivative, central: $f_i' = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$

2nd derivative, central: $f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2)$



This notation indicates how the error in the approximation scales with the size of the mesh

Tutorial question

Try Q1 on the tutorial sheet (**approximating the derivatives from a discrete function**).

- 1. The value of the function $f(x)$ is known in the following three points

$$f(1) = 1, \quad f(1.2) = 1.2, \quad f(1.4) = 1.15$$

(a) Using central differences, determine the approximate value of $f'(1.2)$ and $f''(1.2)$.

(b) Using forward differences, determine the first-order approximation for $f'(1)$.

(c) Using backward differences, determine the first-order approximation for $f'(1.4)$.

Forward: $f_i' = \frac{f_{i+1} - f_i}{h}$ Central: $f_i' = \frac{f_{i+1} - f_{i-1}}{2h}$

Backward: $f_i' = \frac{f_i - f_{i-1}}{h}$ Central: $f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$

Tutorial question

Try Q4 on the tutorial sheet (**central differences with boundary values**).

- 4. It is known that $f''(x) = 1$ for all $x \in [0, 1]$. Additionally, we set $f(0) = 1$ and $f(1) = 0$.

  (a) Use central differences to determine the approximate value of $f(0.5)$.

  (b) Use uniform mesh to discretize the interval $[0, 1]$ with 3 discretization intervals. Use central difference to determine $f(1/3)$ and $f(2/3)$.

  (c) This is an example of the so-called boundary value problem. Integrate the equation $f''(x) = 1$ twice and determine the unknown constants using $f(0) = 1$ and $f(1) = 0$. Compare the exact solution with your approximations from parts (a) and (b).

Central difference approximation: $f_i'' = \dfrac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$

Tutorial question

Try Q3 on the tutorial sheet (**if time**).

- 3. Let $f(x) = x^3$. Determine the error of the forward difference approximation to $f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h}$ evalutated at some arbitrary x_0. Discuss the dependence of the error on x_0 and h. Now derive the error term for the central difference approximation to $f'(x_0)$ and compare the results.

For forward difference: compare $\frac{(x_0+h)^3 - x_0^3}{h}$ with $\frac{d}{dx}(x^3)$

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That's all for today!