Module 1: Matrices & Eigenvalues

Week 4 Tutorial

Matrix methods for differential equations

Key goals for the class

1. How do we convert a system of **coupled 1st order differential equations** into a matrix equation?

2. How do we convert a **higher-order differential equation** into a system of 1st order equations?

3. How do we solve these differential equations given **initial values**?

Coupled differential equations

Consider two variables $x(t)$ and $y(t)$ linked by:

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

These are **coupled differential equations** since $x(t)$ and $y(t)$ are inter-twined. Can be written in matrix form as:

$$\frac{d\vec{x}}{dt} = \mathbf{A}\,\vec{x} \qquad \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If we try a solution $\vec{x}(t) = \vec{v}\,e^{\lambda t}$ we find $\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$ – our differential equation has become an **eigenvalue problem**!

Method for solving…

1. Write the coupled differential equation as a matrix equation, $\frac{d\vec{x}}{dt} = \mathbf{A}\,\vec{x}$

2. Assume a solution $\vec{x}(t) = \vec{v}\,e^{\lambda t}$, so $\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$

3. Find the eigenvalues and eigenvectors of $\mathbf{A}$

4. The solution to the differential equation is then $\vec{x}(t) = C_1\,\vec{v}_1\,e^{\lambda_1 t} + C_2\,\vec{v}_2\,e^{\lambda_2 t}$ (C_1, C_2 constants)

5. Use the initial conditions to find C_1 and C_2

Tutorial question

Try Q1 on the tutorial sheet (**coupled differential equations**).

- 1. Consider the system of coupled differential equations

$$\begin{aligned}\frac{dx}{dt} &= y - x,\\ \frac{dy}{dt} &= x - y.\end{aligned}$$

(a) Give the general solution for arbitrary initial conditions.

(b) Determine the specific solution of the initial value problem with $x(0) = 1$ and $y(0) = 0$.

(c) Determine the solution in the limit of $t \to \infty$.

Higher-order differential equations

Consider a 2nd order differential equation for $x(t)$:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

Let's **introduce a variable** $y = \frac{dx}{dt}$. We can then convert the equation into two 1st order differential equations:

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -bx - ay$$

which may be converted into a **matrix equation** as before!

We can generalize the method for **higher-order** equations

Tutorial question

Try Q2 on the tutorial sheet (**expressing a 2nd order differential equation as two 1st order equations**).

- 2. Consider the second-order differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + \frac{5}{4}y = 0.$$

(a) Convert the above second-order differential equation into a system of first order equations.

(b) Give the general solution.

(c) Determine the specific solution that satisfies the following initial conditions $y(2) = 1$ and $dy(2)/dt = 0$.

Tutorial question

Try Q3 on the tutorial sheet (**generalizing the method for higher-order differential equations**).

- 3. Convert

$$y^{(5)}(t) + y(t) = 0,$$

where $y^{(5)}(t)$ stands for the fifth derivative, into a system of five first-order equations.

Hint: Let $x_1 = y,\ x_2 = \frac{dy}{dt},\ x_3 = \frac{d^2y}{dt^2},\ x_4 = \frac{d^3y}{dt^3},\ x_5 = \frac{d^4y}{dt^4}$ and see if you can write an equation $\frac{d\vec{x}(t)}{dt} = \mathbf{A}\,\vec{x}(t)$

Tutorial question

Try Q4 on the tutorial sheet (**if time**).

- 4. * The centre of mass of a massive concrete pole has the coordinates (x, y, z). In equilibrium $x = y = z = 0$. However, if the pole is deformed (e.g. due to strong wind), the coordinates (x, y, z) change in time t and obey the following equations of motion

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & -2 & -1 \\ -1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}.$$

As the chief engineer of the project, should you be worried about the structural integrity of the pole due to resonance? Justify you answer.

Hint: If we assume $\vec{x}(t) = \vec{c}\, e^{\lambda t}$, where $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, what values of λ are allowed?

That’s all for today!