Module 1: Matrices & Eigenvalues

Week 3 Tutorial

Matrix diagonalization

Key goals for the class

1. How do we **diagonalize** a matrix and why is this useful?
2. How can we **reconstruct a matrix** knowing its eigenvalues and eigenvectors?
3. What are the special properties of **symmetric matrices**?
4. How do we represent a **quadratic curve** using a matrix equation, and determine what sort of curve it represents?

Diagonal matrices

A **diagonal matrix** only contains non-zero entries in the diagonal elements:

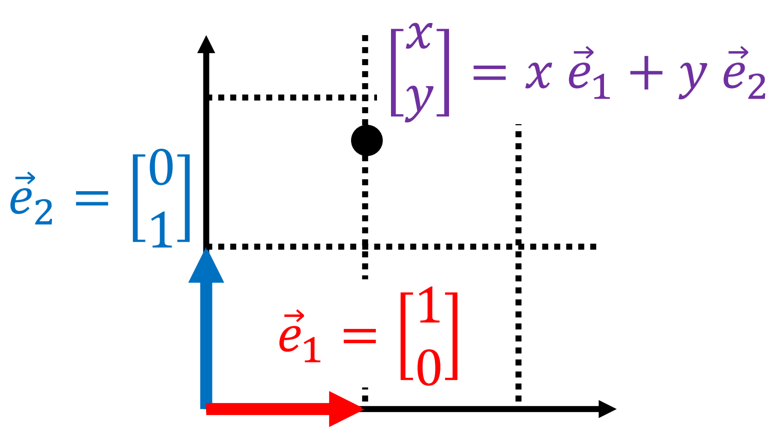
Diagonal matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

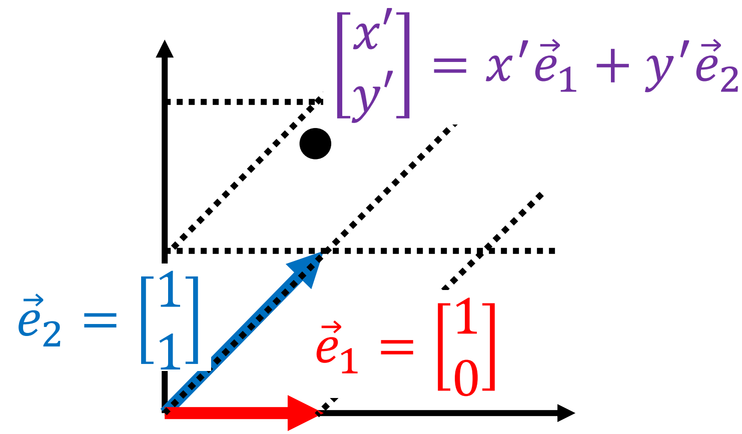
Diagonal matrices are very nice to use for matrix operations like multiplications or inverses!

Basis vectors

Basis vectors are linearly independent vectors that span our co-ordinate space, for example:



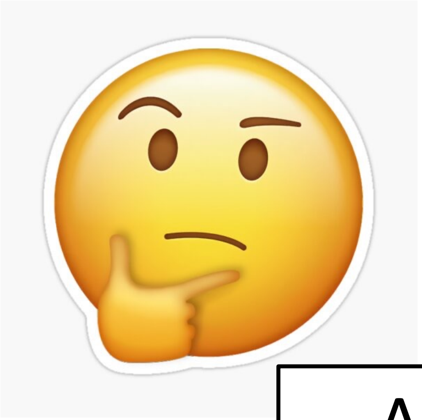
We may choose **another set** of basis vectors, for example:



The **change of basis matrix** $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ transforms the co-ordinates of a point between the different bases: $\vec{x} = \mathbf{B}\,\vec{x}'$

A **linear transformation** $\mathbf{A}$ in one basis can be expressed in the co-ordinate system of the new basis as $\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{B}$

What is matrix diagonalization?



A matrix represents a **linear transformation**

Diagonalizing a matrix means **finding the basis in which this transformation is a diagonal matrix**

This basis is given by the **eigenvectors**, and the transformation is a scaling by the **eigenvalues**

Key formula for diagonalization

If $\mathbf{A}$ is an $n \times n$ matrix with an eigenvalue **spectrum** $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then

$$\mathbf{M}^{-1}\mathbf{A}\,\mathbf{M} = \mathbf{\Lambda}$$

Modal matrix $\mathbf{M} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_n \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

Spectral matrix $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

- To **reconstruct a matrix** from its eigen-things: $\mathbf{A} = \mathbf{M}\,\mathbf{\Lambda}\,\mathbf{M}^{-1}$
- If $\mathbf{A}$ is a **symmetric matrix** (see below), then $\mathbf{M}^{-1} = \mathbf{M}^{T}$
- $\mathbf{A}$ is diagonalizable if its eigenvectors are **linearly independent**

Tutorial question

Try Q1 on the tutorial sheet (**practising the matrix diagonalization formula**).

- 1. In Tutorial 2 we have shown that the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

are given by $\lambda = 2, 3$. The corresponding eigenvectors are

$$\boldsymbol{v}_{(\lambda=2)} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ t \in (-\infty, \infty) \qquad \boldsymbol{v}_{(\lambda=3)} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ s \in (-\infty, \infty).$$

(a) Write down the modal matrix M.

(b) Find the inverse M^{-1}.

(c) Show that

$$M^{-1}AM = \Lambda,$$

where Λ is the spectral matrix.

Symmetric matrices

What is a **symmetric matrix**?

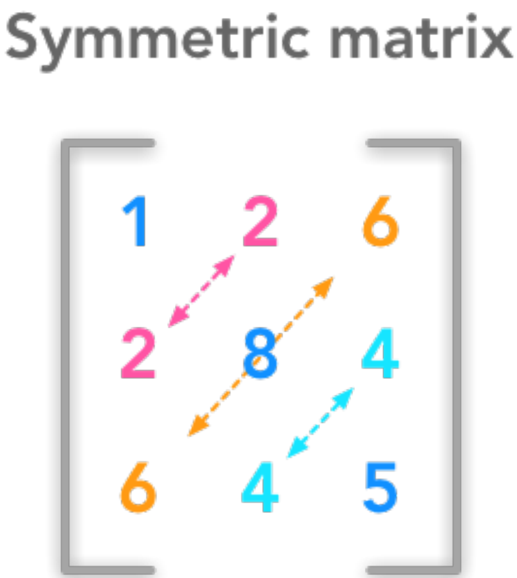
What is special about the eigenvalues and eigenvectors of a symmetric matrix?

Symmetric matrices

What is a **symmetric matrix**?

Transpose is equal to the original matrix!

$$\mathbf{A}^T = \mathbf{A}$$



Special properties of symmetric matrices:

1. Eigenvalues are **real numbers** (not complex)
2. Eigenvectors are **orthogonal** (means that $\mathbf{M}^{-1} = \mathbf{M}^T$)
3. An $n \times n$ symmetric matrix always has **n eigenvectors**

Tutorial question

Try Q2 on the tutorial sheet (**symmetric matrix example**).

- 2. The eigenvectors of the symmetric matrix

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

are given by

$$\boldsymbol{v}_1 = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{v}_3 = p\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \; t, s, p \in (-\infty, \infty).$$

(a) Normalize the eigenvectors and write down the modal matrix M.

(b) Show that the eigenvectors that correspond to distinct eigenvalues are orthogonal.

(c) Calculate

$$M^T AM$$

and thus, determine all eigenvalues of A.

Quadratic curves

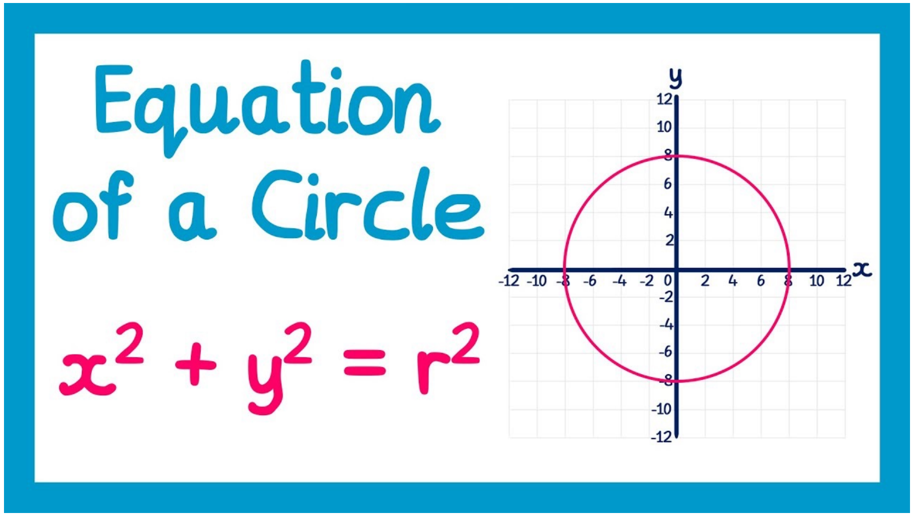
A **quadratic curve** (in 2D) has an equation of the form

$$ax^2 + by^2 + 2cxy = 1$$

It can be expressed in **matrix form** by writing

$$\vec{x}^T \mathbf{A} \, \vec{x} = 1$$

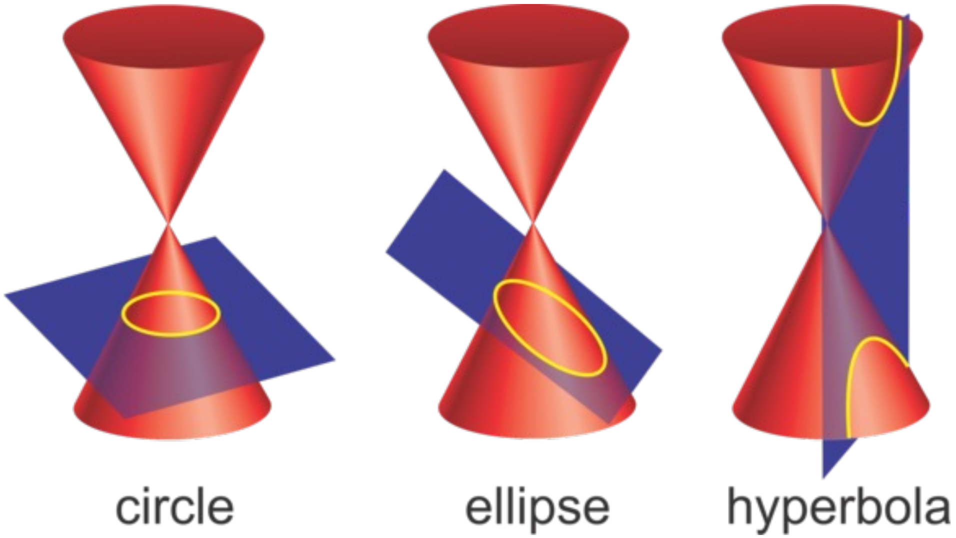
where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$



The type of surface can be found from the matrix $\mathbf{A}$

Quadratic curves

What other types of curve can "$ax^2 + by^2 + 2cxy = 1$" represent?



Steps to classify a quadratic curve

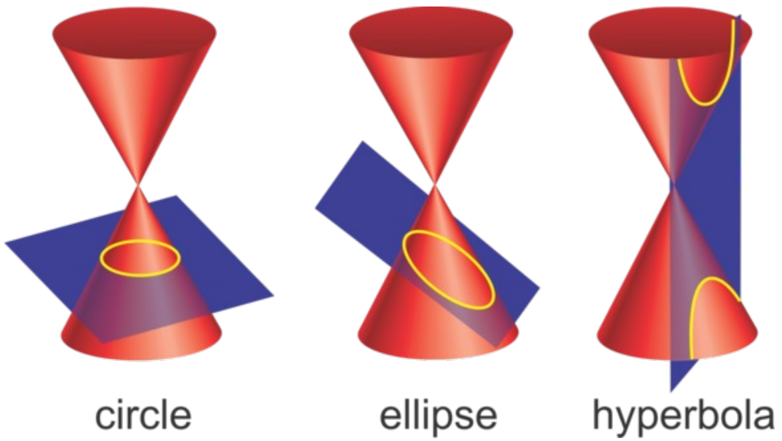
1. Express the equation of the curve in the form $\vec{x}^T \mathbf{A} \, \vec{x} = 1$
2. Find the **eigenvalues** and **unit eigenvectors** of $\mathbf{A}$
3. Construct the modal matrix $\mathbf{M}$ (of unit eigenvectors) and spectral matrix $\boldsymbol{\Lambda}$ (of eigenvalues λ)
4. The surface can now be written in **canonical form** as $\vec{y}^T \boldsymbol{\Lambda} \, \vec{y} = 1$, where the **principal axes** are given by $\vec{y} = \mathbf{M}^T \, \vec{x}$

Classifications in 2D:

λ's the same: **circle**

λ's the same sign: **ellipse**

λ's opposite signs: **hyperbola**



Tutorial question

Try Q3 on the tutorial sheet (**quadratic curve**).

3. Identify the type of the quadratic curve given by

$$5x^2 - 2\sqrt{3}xy + 7y^2 = c$$

where c is a constant.

Does the type of the curve depend on $c > 0$?

What happens if $c < 0$?

Tutorial question

Try Q4 on the tutorial sheet (**if time**).

- 4. (a) Draw the curve from Q3 and determine the directions of the axes of symmetry.

  (b) There exists an orthogonal linear transformation of coordinates (x, y)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

  that transforms the quadratic curve in Q3 into its canonical form. Determine the matrix $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and write down the quadratic curve in the new coordinates (x', y'). Give geometric interpretation for the transformation used.

  (c) Discuss the difference between two similarity transformations that we use to diagonalize a matrix A: $M^{-1}AM = \Lambda$ and $M^{T}AM = \Lambda$.

That's all for today!