Module 1: Matrices & Eigenvalues

Week 2 Tutorial

Eigenvalues and eigenvectors

Key goals for the class

1. What is the meaning of the **eigenvectors** and **eigenvalues** of a matrix?

2. How do we **determine** eigenvectors and eigenvalues?

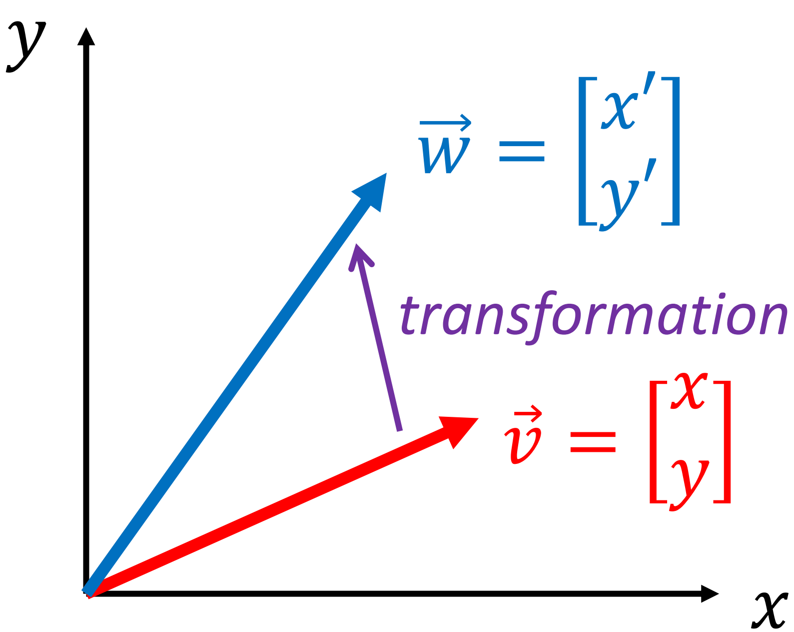
3. What are the relations between eigenvalues and the **determinant** and **trace** of a matrix?

4. **How many** eigenvectors does a matrix have?

Matrices are linear transformations!

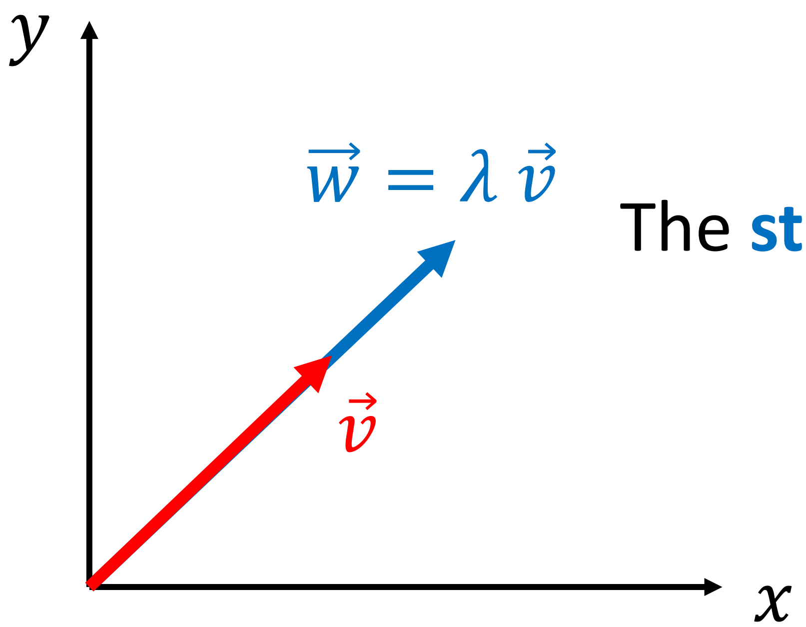
Matrices can be thought of as **linear transformations** which **rotate** and **stretch** a vector:

$$\vec{w} = \mathbf{A}\,\vec{v} \quad \rightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



What are eigen-things?

An **eigenvector** of a matrix is the special direction where the linear transformation corresponds to **only a stretching**



$$\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$$

The **stretch factor** λ is the **eigenvalue**

Typically, there are 2 such special directions in 2D space (… and 3 special directions in 3D space …)

The eigenvectors and eigenvalues geometrically represent the properties of the linear transformation

How do we find the eigen-things?

How do we find the eigenvectors and eigenvalues of a matrix, that satisfy

$$\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$$

How do we find the eigen-things?

How do we find the eigenvectors and eigenvalues of a matrix, that satisfy $\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$?

Re-write as …

$$\mathbf{A}\,\vec{v} = \lambda\,\mathbf{I}\,\vec{v} \quad \rightarrow \quad (\mathbf{A} - \lambda\,\mathbf{I})\,\vec{v} = 0$$

The only way this can be true is if $|\mathbf{A} - \lambda\,\mathbf{I}| = 0$

(otherwise, we could find a matrix inverse and would just have $\vec{v} = 0$)

Example: what are the eigenvalues of $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$?

We need to solve the equation $\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

How do we find the eigen-things?

How do we find the eigenvectors and eigenvalues of a matrix, that satisfy $\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$?

The general process is:

1. Find the **eigenvalues** by solving $|\mathbf{A} - \lambda\,\mathbf{I}| = 0$
   (where $\boldsymbol{I}$ is the identity matrix)

2. For each eigenvalue λ, find the corresponding **eigenvector** by substituting in $\mathbf{A}\,\vec{v} = \lambda\,\vec{v}$ and using one of the rows
   (the other row will give the same answer)

3. If needed, **normalize** the eigenvector such that $\vec{v} \cdot \vec{v} = 1$
   (or else can leave in terms of a scalar variable)

Tutorial question

Try Q1 on the tutorial sheet (**determining eigenvalues and eigenvectors of a 2x2 matrix**).

- 1. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

Confirm that the eigenvectors are linearly independent.

Properties of eigenvalues

The eigenvalues of a matrix $\mathbf{A}$ have two useful properties:

1. The **product** of the eigenvalues equals the determinant of $\mathbf{A}$

$$\lambda_1 \lambda_2 \cdots \lambda_n = |\mathbf{A}|$$

2. The **sum** of the eigenvalues equals the trace of $\mathbf{A}$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{Tr}(\mathbf{A})$$

These properties can be used to determine eigenvalues more easily, if some are already known

Tutorial question

Try Q2 on the tutorial sheet (**properties of eigenvalues**).

- 2. One of the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

is $\lambda = 2$.

(a) Use the properties of the eigenvalues to find all other eigenvalues of A

$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{Tr}(A) = a_{11} + a_{22} + a_{33}, \quad det(A) = \lambda_1 \lambda_2 \lambda_3.$$

(b) Find the corresponding eigenvectors.

How many eigenvectors?

Do all 2×2 matrices have 2 eigenvectors?

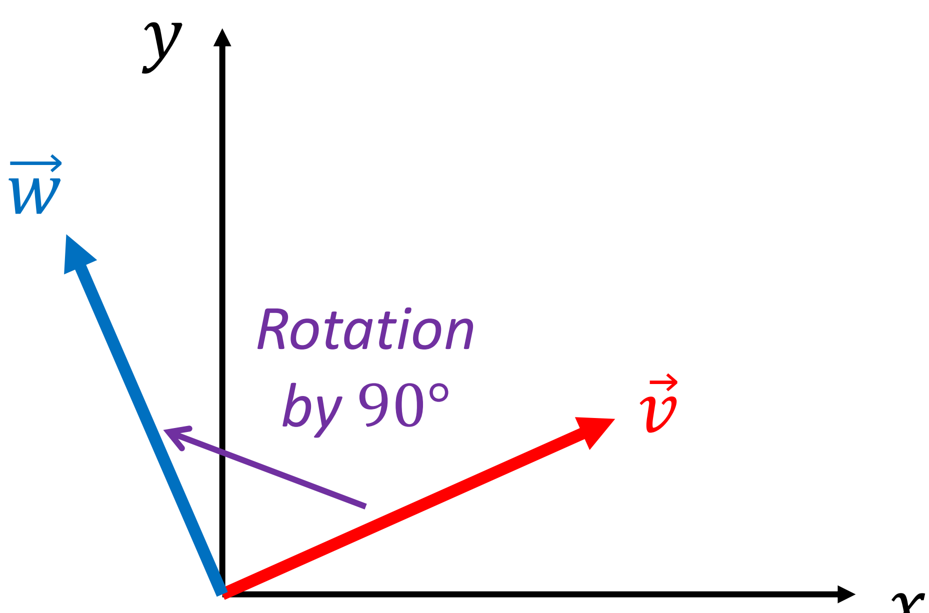
If not, how many eigenvectors can they have?

How can we tell?

How many eigenvectors?

Do all 2×2 matrices have 2 eigenvectors?

Example 1: Consider a linear transformation $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which corresponds to a rotation by 90° at all positions



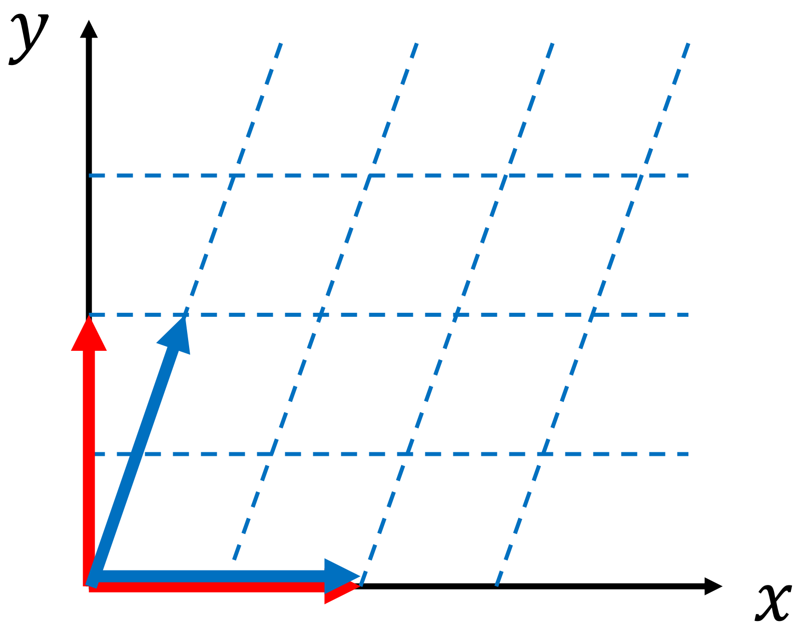
There are clearly **no** vectors which would only be stretched by this transformation, so **there are no eigenvectors**

Applying $|\mathbf{A} - \lambda\,\mathbf{I}| = 0$ gives $\lambda^2 = -1$ so $\lambda = \pm i$ – no real number solutions!

How many eigenvectors?

Do all 2×2 matrices have 2 eigenvectors?

Example 2: Consider a shear $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which tilts the y-axis but leaves the x-axis unchanged:



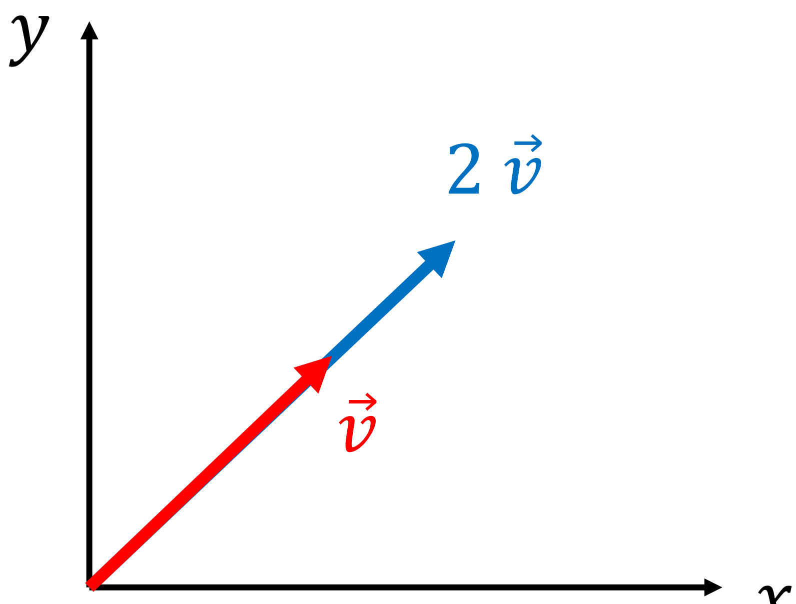
The only vector which is not rotated lies along the x-axis, so **there is only 1 eigenvector**

Applying $|\mathbf{A} - \lambda\,\mathbf{I}| = 0$ gives $(1-\lambda)^2 = 0$ so $\lambda = 1$ is the only solution

How many eigenvectors?

Do all 2×2 matrices have 2 eigenvectors?

Example 3: Consider a matrix $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, which multiplies every vector by 2!



Every vector is an eigenvector with eigenvalue 2 – there are an **infinite number of eigenvectors** in this case!

Applying $|\mathbf{A} - \lambda\,\mathbf{I}| = 0$ gives $(2-\lambda)^2 = 0$ so $\lambda = 2$ is the only solution

Tutorial question

Try Q3 and Q4 on the tutorial sheet (**properties of eigenvectors**).

- 3. The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

are $\lambda = 1, 1, 0$. Find the corresponding eigenvectors. Is it possible to find any three linearly independent eigenvectors?

- 4. Given that $\lambda = 1$ is a two-times repeated eigenvalue of the matrix

$$\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

find the corresponding eigenvector.

Note: if eigenvalues are repeated, n linearly independent eigenvectors cannot always be found

Tutorial question

Try Q5 on the tutorial sheet (**if time**).

- 5. Prove the following statements.

  (a) If $\vec{v}$ is an eigenvector of A with the eigenvalue λ, then $\vec{v}$ is also an eigenvector of A^{-1} with the eigenvalue $1/\lambda$.

  (b) The spectrum of A is identical to the spectrum of A^T.

  (c) If one of the eigenvalues of A is $\lambda = 0$, then A is not invertible.

  (d) For a 2×2 matrix A, the eigenvalues are given by

$$\lambda = \frac{\mathrm{Tr}(A)}{2} \pm \frac{\sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}}{2}.$$

That’s all for today!