Module 3: Probability and Statistics

Week 12 Tutorial

Hypothesis tests

Key goals for the class

1. How do we apply a **hypothesis test for the population mean**, given a data sample?

2. How do we check whether binned data is correlated, using a χ^2 **hypothesis test**?

3. How do we check whether strings of data values are correlated, using a **correlation coefficient hypothesis test**?

Hypothesis tests

Hypothesis testing is a common feature of "frequentist statistics"

1. State **null hypothesis** (H_0) and alternate hypothesis (H_1)

2. Determine the **test statistic**

3. Specify its **critical value** for the chosen significance level

4. State the **rejection rule** for H_0

5. Calculate the **observed value** of the test statistic

6. Draw the **conclusion**

Hypothesis test for the mean

Sample mean and size

Population standard deviation is known

"If the mean of a sample of $N = 16$ is 3.42, and the population standard deviation is 0.68 – test, at significance level 5%, whether population mean < 3.70"

Significance level

Range of population mean

1. **Null hypothesis** H_0 is $\mu = 3.70$, H_1 is $\mu < 3.70$

2. Test statistic is **z-score**, since the standard deviation is known

3. **Critical value** is $z_c = -1.645$ for 5% significance $\rightarrow$ rejection rule for H_0 is $z < -1.645$

4. **Observed value** is $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{N}}$ $\rightarrow$ carry out hypothesis test

Reading the normal distribution table

Normal Distribution

Distribution function

The table gives probability $P = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$.
For $x < 0$ values of $\Phi(x)$ can be obtained from $\Phi(-x) = 1 - \Phi(x)$.

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

The z-score maps to the x value in the normal distribution table

If $P = 0.95$ (5% probability in the tail), then $z = 1.645$

Tutorial question

Try Q1 on the tutorial sheet (**testing a hypothesis for the mean, knowing the population standard deviation**).

- 1. The mean weight loss of 16 grinding balls after a certain length of time in mill slurry is 3.42 g Assume that the weight loss of individual grinding balls is normally distributed and that the population standard deviation is 0.68 g.

  (a) Test, at 5% significance level, whether the population mean weight loss is less than 3.70 g.

  (b) You are looking to prove that the population mean weight loss is different from 3 g. Determine the p-value, in this case.

Hypothesis test for the mean

If the population standard deviation is **not known**, and must be estimated from the sample, the steps change.

Question: Given a data sample of size N, test at 2.5% significance whether the population mean $\mu > 23$

1. Compute the sample mean $\bar{x}$ and standard deviation s

2. **Null hypothesis** H_0 is $\mu = 23$, H_1 is $\mu > 23$

3. Test statistic is t-**score**, since the standard deviation is unknown

4. **Critical value** is $t_c = 2.365$ (using $d.f. = N - 1$ and $p = 0.025$)

5. **Observed value** is $t = \dfrac{\bar{x} - \mu}{s/\sqrt{N}}$ $\rightarrow$ carry out hypothesis test

Reading the t-distribution table

1-tailed probability ⟶

Degrees of freedom ($= N - 1$ for hypothesis test of mean)

Example: for 2.5% significance test for a sample $N = 8$, critical value is $t_c = 2.365$

t distribution critical values

df	\.25	\.20	\.15	\.10	\.05	\.025	\.02	\.01	\.005	\.0025	\.001	\.0005
						Upper tail probability p						
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
z^*	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%	99.9%
						Confidence level C						
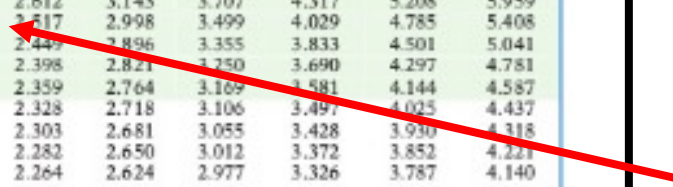
Tutorial question

Try Q2 on the tutorial sheet (**testing a hypothesis for the mean, not knowing the population std.dev.**).

- 2. School of Australian salmon is spotted off Gunnammata beach. Fishermen catch and measure the length of 8 fish in cm

$$\{22, 23, 25, 25, 24.5, 26, 27, 23\}$$

Assuming that the length of the fish is normally distributed, test at $\alpha = 2.5\%$ significance that the mean length of Australian salmon in the school is larger than 23 cm.

Correlation testing

We now wish to check whether two random variables are **correlated** (vary together) or **independent**. We'll discuss:

1. χ^2 ("chi-squared") hypothesis test on contingency table

2. Correlation coefficient hypothesis test using values
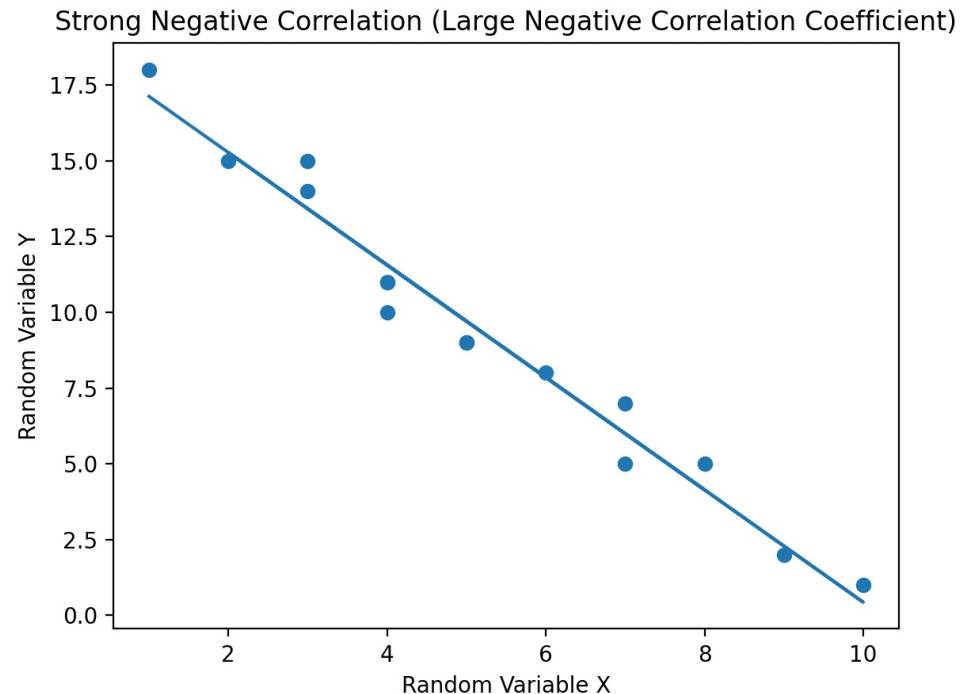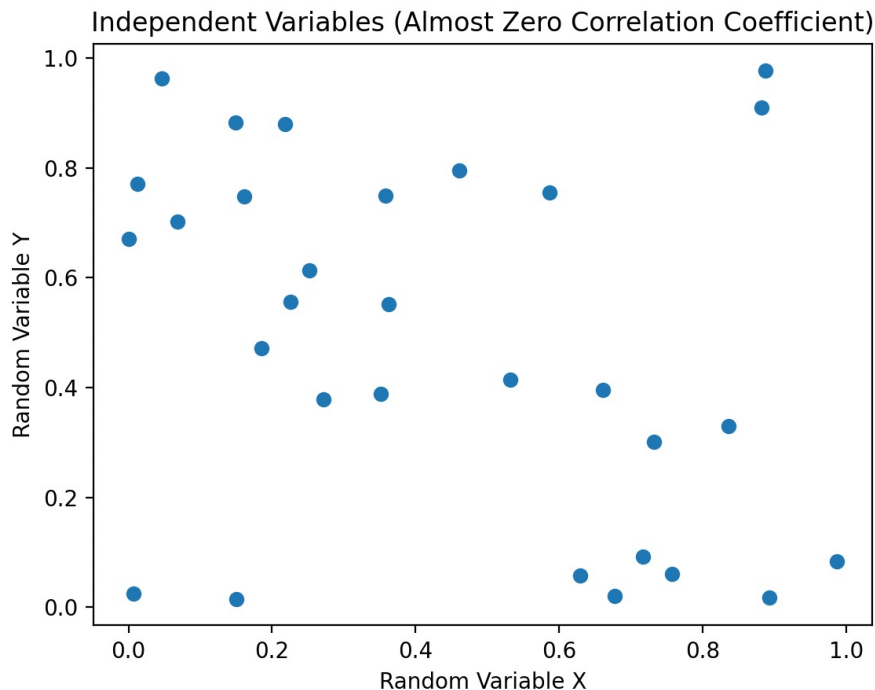
χ^2 test for independence

The first correlation test uses a 2D data table such as:

Table 1: Weekend sales.

Type of shop	Saturday	Sunday	
Shop 1	100	30	130
Shop 2	12	10	22
Shop 3	34	15	49
Shop 4	7	10	17
	153	65	218

Is there a relation between the day and the shop activity?

1. Write the **row and column totals** ("marginal distributions")

2. Use these to create the **expected frequencies** in each cell assuming the variables are independent, $E = \dfrac{\text{row total} \times \text{column total}}{\text{grand total}}$

3. Determine $\chi^2 = \sum \dfrac{(O-E)^2}{E}$ where $O = \text{observed}$, $E = \text{expected}$

4. Compare χ^2 to **test threshold**, using $d.f. = (N_{\text{row}} - 1) \times (N_{\text{col}} - 1)$

Reading the χ^2 table

Probability in the tail ⟶

Degrees of freedom
$(= (N_{\text{row}} - 1) \times (N_{\text{col}} - 1)$
for independence test)

Table C χ^2 critical values

Tail probability p

df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4

Example: for 98% confidence if $d.f. = 3$, we need $\chi^2 > 9.84$

If $\chi^2 > 9.84$, then the **null hypothesis of independent variables is rejected**

Tutorial question

Try Q3 on the tutorial sheet (χ^2 **test for the independence of variables**).

3. Four different internet shops counted the number of sales on a given weekend (Saturday and Sunday). Is the type of the shop independent of the day?

Table 1: Weekend sales.

Type of shop	Saturday	Sunday
Shop 1	100	30
Shop 2	12	10
Shop 3	34	15
Shop 4	7	10

(c) Conduct the χ^2 test of independence and conclude at 2% significance.

(d) Conduct the χ^2 test of independence and conclude at 0.5% significance.

Testing for a linear relationship

The second correlation test searches for a **linear relationship** between strings of data such as:

$$x = \{\ 2.1,\ 2.1,\ 2.1,\ 2.2,\ 2.3,\ 2.4,\ 2.4,\ 2.4,\ 2.5,\ 2.5\ \}$$

$$y = \{\ 43,\ 44,\ 45,\ 49,\ 55,\ 58,\ 56,\ 51,\ 52,\ 62\ \}$$

Is there a linear relation between x and y?

1. Find the **Pearson coefficient** $r = \dfrac{\sum(x_i - \bar{x})(y_i - \bar{y})}{(N-1)\ s_x\ s_y}$

   (you can use the green calculator to do this if you wish)

2. Compute the test statistic $\boldsymbol{t} = \dfrac{r\sqrt{N-2}}{\sqrt{1-r^2}}$

3. Compare t to **test threshold**, using $d.f. = N - 2$

Tutorial question

Try Q4 on the tutorial sheet (**testing for a linear relationship between two variables**).

4. Consider two sets of data. **Data source:** Ember (2025); Energy Institute - Statistical Review of World Energy (2025).

Nuclear power production (world, thousands of TWh, 1991-2005):

{2.1, 2.1, 2.1, 2.2, 2.3, 2.4, 2.4, 2.4, 2.5, 2.5}.

Number of Australian student visas issues (thousands) to commence in 2005-2014 (Project 2025, S2):

{43, 44, 45, 49, 55, 58, 56, 51, 52, 62}.

Is there a linear relation between these two variables? Give as many details as possible.

That's all for today!