Module 1: Matrices & Eigenvalues

Week 1 Tutorial

Matrices and systems of equations

Hello!

- I'm **Chris Blake** and I'm a lecturer in the Centre for Astrophysics & Supercomputing
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- E-mail: cblake@swin.edu.au
- Use Canvas inbox/messaging
- Chat to me during tutorials/labs

Tutorial question sheet

▾ Week 1

- MATLAB resources
- Study plan for Week 1 (August 4 -August 10)
- Tutorial 1
- MATLAB Laboratory 1
- lecture notes from study guide

Key goals for the class

1. How do we evaluate the **determinant**, **trace** and **inverse** of a matrix?

2. How do we use matrices to solve systems of linear equations, applying **Gaussian elimination** or **Cramer's rule**?

3. What is the significance of vectors being **linearly independent**, and how can we test for it?

Determinant and trace

What is the **determinant** and **trace** of:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant and trace

The **determinant** of a 2×2 matrix $\mathbf{A}$ can be evaluated as:

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The **determinant** of a 3×3 matrix $\mathbf{B}$ can be evaluated as:

$$|\mathbf{B}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The **trace** of a matrix is the sum of its diagonal elements:
$\mathrm{Tr}(\mathbf{A}) = a + d$ and $\mathrm{Tr}(\mathbf{B}) = a + e + i$

Inverse

The **inverse** $\mathbf{A}^{-1}$ of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix (for example in 2D, $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

What type of question/problem would we solve using a matrix inverse?

Inverse

The **inverse** $\mathbf{A}^{-1}$ of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix (for example in 2D, $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Inverse matrices are useful when we need to solve a **system of linear equations** such as:

$$2x + 3y = 8$$
$$4x + y = 6$$

$$\rightarrow \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{A} \quad \mathbf{z} \quad = \quad \mathbf{r}$$

$$\rightarrow \quad \mathbf{z} = \mathbf{A}^{-1}\mathbf{r}$$

Coefficient matrix

Vector of unknowns $\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}$

Vector of right-hand sides

Inverse matrix

Inverting a matrix by hand

2×2 matrix: easiest to use the formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Do all matrices have inverses?

Inverting a matrix by hand

2×2 **matrix**: easiest to use the formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The matrix inverse does not exist if its determinant is zero!

3×3 **matrix**: many methods available

For example, row operations:

Find inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix} \rightarrow$ construct "double" matrix $\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & -2 & 0 & 1 & 0 \\ 3 & 4 & -1 & 0 & 0 & 1 \end{array}\right)$

Perform matrix row operations to convert the left-hand 3x3 matrix to the identity matrix, the right-hand 3x3 matrix is then the matrix inverse!

Solving a system of equations

Matrices are very useful for solving **systems of linear equations**. For example:

$$\begin{aligned} x + 2y + 2z &= 3 \\ 2x + 2y - 2z &= 1 \\ 3x + 4y - z &= 3 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

What methods are available?

Gaussian elimination	Cramer's rule	Matrix inverse

Gaussian elimination

Gaussian elimination is typically the easiest method for solving a matrix equation

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

Augmented matrix $\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 2 & 2 & -2 & 1 \\ 3 & 4 & -1 & 3 \end{array}\right)$

Apply matrix row operations until the augmented matrix has three zeros in the bottom left-hand corner, then read off the new equations

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ \mathbf{0} & -2 & -6 & -5 \\ \mathbf{0} & \mathbf{0} & -1 & -1 \end{array}\right) \rightarrow \begin{aligned} x + 2y + 2z &= 3 \\ -2y - 6z &= -5 \\ -z &= -1 \end{aligned}$$

… then use back substitution

Cramer's rule

Cramer's rule is another possible method for solving a matrix equation:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix}$$

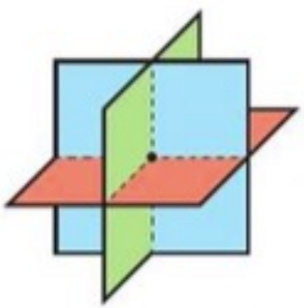
$$\mathbf{A}_x = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & -2 \\ 3 & 4 & -1 \end{bmatrix} \qquad \mathbf{A}_y = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -2 \\ 3 & 3 & -1 \end{bmatrix} \qquad \mathbf{A}_z = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix}$$

$$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} \qquad y = \frac{|\mathbf{A}_y|}{|\mathbf{A}|} \qquad z = \frac{|\mathbf{A}_z|}{|\mathbf{A}|}$$

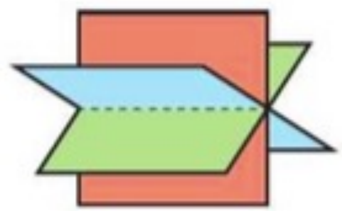
Geometrical meaning of the solution

An equation like $2x + 3y = 8$ represents a **line in 2D space** → Solving 2 simultaneous equations for 2 unknowns is like ***solving where 2 lines intersect***

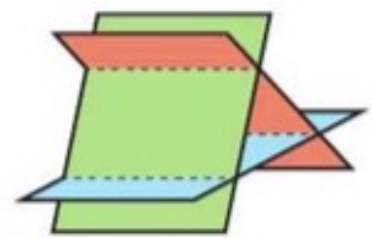
An equation like $x + 2y + 2z = 3$ represents a **plane in 3D space** → Solving 3 simultaneous equations for 3 unknowns is like ***solving where 3 planes intersect***



The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.



The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.

Tutorial question

Try Q1(a)(i) and Q2(i) on the tutorial sheet (**solutions of linear systems of equations**)

Solve the following systems of equations by using Gaussian elimination and state the geometrical meaning of the solutions:

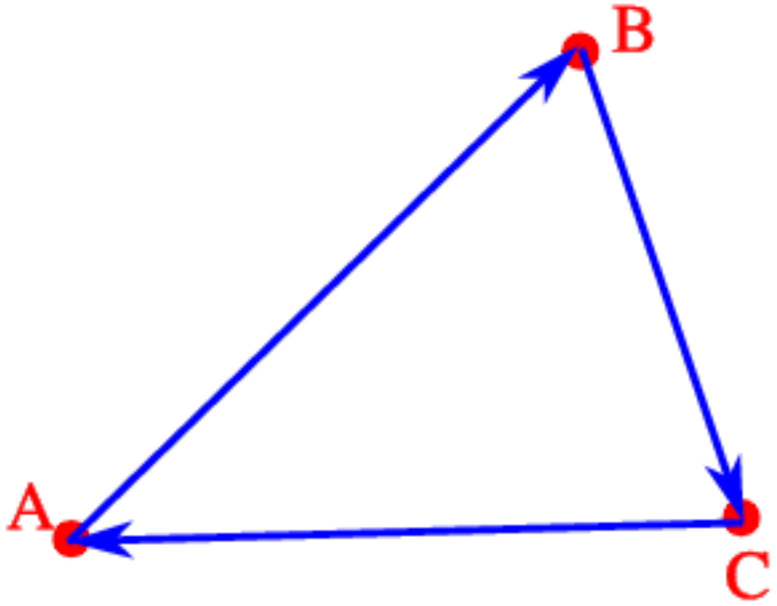
Q1: $$\begin{cases} x + 2y + 2z = 3 \\ 2x + 2y - 2z = 1 \\ 3x + 4y - z = 3 \end{cases}$$

Q2: $$\begin{cases} x + 2y + 2z = 3 \\ 2x + 2y - 2z = 0 \\ 3x + 4y = 3 \end{cases}$$

(If you finish this, try the second example given in each case.)

Linear (in)dependence of vectors

What does it mean if a set of vectors is "linearly dependent"?



Linear (in)dependence of vectors

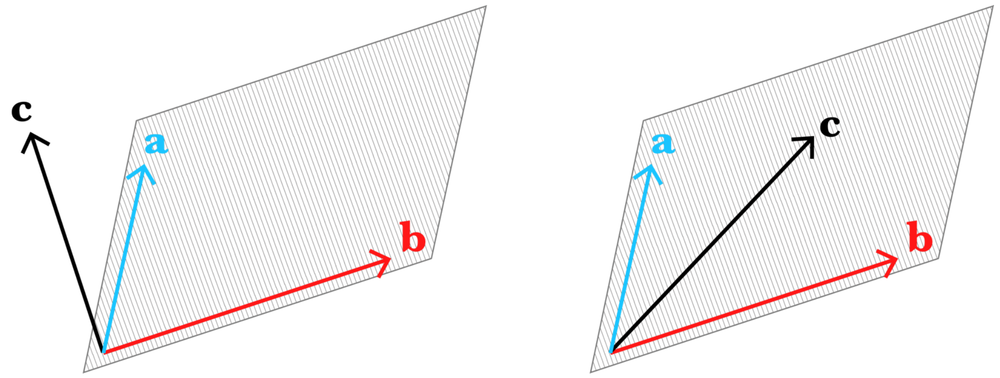
A set of vectors is **linearly dependent** if …

… one vector is a scaled sum of the other vectors

… (or equivalently) a scaled sum of the vectors is zero

Otherwise, the set of vectors is **linearly independent**!

*This determines the space that can spanned by a **linear combination** of the vectors. It requires three linearly-independent vectors to span 3D-space.*



Testing for linear (in)dependence

To **test whether a set of vectors is linearly dependent**, form a matrix out of the vectors, and check whether its determinant is zero:

e.g. are $\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ a linearly independent set?

Consider $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \;\; \rightarrow$ linear independence

Doesn't matter if writing as rows or columns, since $|\mathbf{A}| = |\mathbf{A}^T|$

This test works because it's establishing if the transformation to this basis, which uses $\mathbf{A}^{-1}$, exists

Tutorial question

Try Q3, Q4, Q5 on the tutorial sheet! (**linear dependence of vectors**)

- 3. Show that the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent.
- 4. Show that the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly dependent.
- 5. Determine for which value of a, the vectros $\vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} -1 \\ -1 \\ a \end{bmatrix}$ are linearly dependent.

Tutorial question

Try Q6 on the tutorial sheet! (**determinants**)

(Try Q7 if you have time.)

- 6. Find the determinants of the following matrices:

$$\text{(a)} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 20 & 0 & 1 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}; \quad \text{(c)} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}.$$

- 7. Invert the above matrices or state why this is impossible.

That's all for today!