Transcript for Video 4: Angular momentum

[1] Welcome to this quantum mechanics video about angular momentum! In the previous video we studied how to solve quantum mechanics problems for particles moving in one dimension. However, one-dimensional cases are quite limited, and we would like to extend these methods to three-dimensional cases, such as the atoms. When moving to three dimensions we can define some new properties of a particle or wavefunction. A key property we'll focus on in this video is angular momentum. Today we're going to discuss the operators representing angular momentum, the eigenvalues of these operators which tell us the values of angular momentum we can observe, and the corresponding eigenfunctions which describe the angular dependence of the wavefunction in three dimensions. Some of this content is more mathematical than before, since it involves the maths of spherical co-ordinates. I'll talk about some of this maths to give you the full picture, but there will be no need for you to reproduce the technical derivations I show.

[2] To start, let's recap the definition of angular momentum. Angular momentum describes motion around an axis, where this axis gives a direction to angular momentum. In this sense, angular momentum is a vector quantity.

[3] The definition of angular momentum about the origin in classical physics is the vector cross product of the position vector of the particle, and its momentum vector. This vector cross product gives the magnitude and direction of the angular momentum of the particle. The magnitude of the angular momentum corresponds to multiplying the momentum of the particle by its perpendicular distance from the axis along which the angular momentum is aligned. The direction is perpendicular to the plane containing the position and momentum vectors.

[4] In quantum mechanics, observables such as angular momentum are associated with operators. We can deduce the operators for angular momentum by substituting in the operators for position and momentum to the same angular momentum formula used in classical physics. Since we're now working in three dimensions, we'll generalise the position operator to a vector consisting of the "x, y, z" co-ordinates, and we'll generalise the momentum operator to a vector which is proportional to the partial derivatives with respect to each of the "x, y, z" co-ordinates, multiplied by "minus i h-bar" as before. If we substitute these forms in the cross product for angular momentum and evaluate the result, we obtain the operators for the angular momentum in the different "x, y, z" directions, which you can see written on the slide as vector components. You can see that these consist of differences between combinations of partial derivatives and co-ordinates.

[5] Before studying these angular momentum operators further, let's recap the meaning of the commutation relation between two operators that we met in the second video. For any pair of operators "A" and "B", we can define the commutation relation as the difference between applying the operators in opposite orders. Because this commutator is a combination of operators, it is itself an operator, so it can be clearer to imagine it acting on a function which we'll call "f". To help interpret this expression, remember for example that the string "A B f" means "first apply operator B to the function f, and then apply operator A to the result". In the previous video we mentioned that if two operators have zero

commutator, it implies that these operators have simultaneous eigenfunctions, and hence the values of the two observables can simultaneously be precisely known.

[6] Let's now check whether the angular momentum operators commute. We'll first define the operator for the total angular momentum, which we'll see is a very useful quantity in quantum mechanics. We can find the square of the total angular momentum, which turns out to be the best way of describing this quantity, by the Pythagoras combination of the different angular momentum components. In this week's activities we'll derive the commutation relations between the different angular momentum components, and I'll just summarise the results here. The operators for the "x" and "y" components of angular momentum do not commute, and the commutator between these operators turns out to be proportional to the operator for the "z" component of angular momentum, multiplied by "i h-bar". I can deduce some equivalent versions of this relation by cycling around the "x, y, z" symbols. We can also prove the second important commutation relation for angular momentum, which is that the operator for the "z" component of angular momentum commutes with the operator for the total angular momentum squared. This result also holds for the operators corresponding to the other angular momentum components.

[7] What do these commutation relations imply? The fact that the operators for the different components of angular momentum do not commute implies that they are not simultaneous observables. In other words, I can only precisely know one angular momentum component at any given time, and the other components remain uncertain. Also, the fact the operators for any component of angular momentum, which we'll take as the "z" component, and the total angular momentum do commute implies that we can precisely know the total angular momentum at the same time as any one component. Since the operators for "L_z" and "L squared" commute, they possess a simultaneous set of eigenfunctions, for which we'll use "capital Y" as the symbol. Here are the eigenfunction-eigenvalue relations for these operators in the most general form – next we'll determine the specific eigenvalues, which for now I've indicated by the Greek letters "lambda" and "mu" in the equation.

[8] We'll sketch out the method used by the textbook for deriving these eigenvalues in the next couple of slides, although you won't need to memorise or reproduce this derivation. We'll use a ladder operator technique, which is similar to the method we applied for stepping between different energy states of the one-dimensional harmonic oscillator. This time, we'll define ladder operators "L plus" and "L minus" to be complex conjugates formed from the operators for the "x" and "y" components of angular momentum. By combining these definitions with the commutation relations for angular momentum we previously derived, we can obtain the additional commutation relations shown on this slide. Please feel free to pause the video and check these relations if you would like to do so.

[9] On the next two slides I've sketched the remaining steps of the derivation of the angular momentum eigenvalues, which again you won't need to memorize. In essence we're determining that these ladder operators convert between the different eigenfunctions of "L_z", which all have the same eigenvalue of "L squared". We're also learning that the eigenvalues of "L_z" are separated by intervals of "h-bar".

[10] Continuing this derivation, we can also form a relation between these ladder operators and the total angular momentum, which we can use to deduce the eigenvalues of "L squared", and from this the maximum and minimum eigenvalues of "L_z". You can find full details of this argument in the textbook, and in this video I will just focus on the final result for the angular momentum eigenvalues. However, the proof is here if you would like to study it.

[11] Let's summarise the results for the angular momentum eigenvalues. We've learned from this derivation that the possible eigenvalues of the total angular momentum "L squared" are "l times l plus 1 times h-bar squared" where "l" is a so-called quantum number of total angular momentum. Given a particular total angular momentum state specified by the value of "l", the possible measurements of the "z" component of angular momentum can range from "minus h-bar l" to "plus h-bar l" in steps of "h-bar". We can equivalently write these eigenvalues as a coefficient "m" multiplied by h-bar, where "m" can range from "minus l" to "plus l". We call "m" the quantum number for the "z"-component of angular momentum. For this arrangement to work, "l" can either be an integer such as 0, 1 or 2, or a half-integer such as one half, three over two, or five over two. You can check that either of these choices would allow us to step over the eigenvalues of "L_z" from the bottom to the top value with the required spacing. Knowing the values of "l" and "m", we will always know the values of "L squared" and "L_z", which are the two simultaneous observables of angular momentum.

[12] The fact that the quantum number "l" for the total angular momentum can either be an integer or a half-integer is pretty interesting, because these choices actually correspond to different types of particles. These two types are called fermions, which have half-integer angular momentum, and bosons, which have integer angular momentum. Here the angular momentum is referring to an intrinsic angular momentum which particles possess, known as spin. You'll learn more about this property in atomic physics!

[13] Before we move on, let's summarise the key points we need to remember about the possible angular momentum eigenvalues. These values are described by two quantum numbers, "l" and "m". The value of "l" determines the total angular momentum, and the value of "m" specifies the "z" component of angular momentum. For a given "l", the value of "m" is restricted to lie in the range from "minus l" to "plus l", separated by integers. These two angular momentum values may be simultaneously known.

[14] Now we've met the eigenvalues, let's consider the corresponding eigenfunctions of angular momentum. These are important because they will help us describe the three-dimensional wavefunction of a particle or atom with given angular momentum values. Since angular momentum involves rotation about an axis, it's convenient to express these eigenfunctions in spherical polar co-ordinates "theta" and "phi", which are projected on a sphere. In spherical polar co-ordinates there is a relation between "theta", "phi" and the radius "r", and the Cartesian "x", "y" and "z" co-ordinates. This is an important diagram so please take a moment to study it, since we'll be referring to the "theta" and "phi" co-ordinates over the next few slides. The "phi" co-ordinate moves us around the equator, like the longitude co-ordinate on the Earth. It ranges between 0 and 360 degrees or "2 pi" in units of radians. The "theta" co-ordinate moves us up and down between the poles, like the

latitude co-ordinate on the Earth. It ranges between 0 and 180 degrees, or “pi” in units of radians.

[15] Let’s start with the eigenfunctions for the “z” component of angular momentum, before we generalise to combine this with the total angular momentum. Previously in this video, we found an equation for the operator for the “z” component of angular momentum, in terms of the “x” and “y” co-ordinates. However, there is a much simpler expression in terms of spherical co-ordinates. By applying the mathematics of the chain rule, and substituting in the relation for the spherical polar co-ordinates, we can show that this combination of the “x” and “y” co-ordinates in the brackets is simply equivalent to the partial derivative with respect to the “phi” co-ordinate, the spherical co-ordinate which rotates us around the “z” axis. Hence, the operator for the “z” component of angular momentum is just “minus i h-bar” multiplied by the derivative with respect to “phi”. You can see how similar this form is to the operator for ordinary linear momentum in one dimension. This is not a coincidence!

[16] The eigenfunctions of the “L_z” operator are hence similar to those we found earlier for linear momentum. We can check by substitution that these eigenfunctions are proportional to “e to the power of i times m times phi”, since when we apply the “L_z” operator to this function, we’ll get back the same function multiplied by an eigenvalue of “m times h-bar”. The value of “m” in the exponent of the eigenfunction has to be an integer – since as I increase the “phi” co-ordinate I’m rotating around a circle, and when I increase “phi” by “2 pi” or 360 degrees, I’ve moved myself fully around the circle back to the same point I started. In this case, the value of the wavefunction must not change, since it represents the same point on the sphere. This condition is automatically satisfied if “m” is an integer, because “e to the power of i times 2 pi” is equal to “1”. The normalisation constant in front of the eigenfunction, that is “1 divided by the square root of 2 pi”, ensures that when I integrate the modulus squared of this eigenfunction across the full range of “phi”, from “0” to “2 pi”, I will obtain an answer of “1”.

[17] The joint eigenfunctions with “L squared” are more complicated because we need to add the additional “theta” co-ordinate! The easiest way to do this is to consider a trial solution for the joint eigenfunctions in which the “e to the power of i m phi” functions, which are the eigenfunctions of “L_z”, are multiplied by a function of theta which I’ve called “P”, as well as a normalisation I’ve called “N”. I’ve labelled the function “P” with subscripts of “l” and “m”, for reasons we’ll see in a moment. Finding the functional form of “P” to complete the eigenfunction is a bit messy, since I need to use the operator for “L squared” in spherical co-ordinates. This operator is derived in the textbooks and I’ll just state its form here. You can see that it involves some complicated derivatives with respect to both spherical co-ordinates, theta and phi!

[18] Substituting in my trial solution for the eigenfunctions and using the known eigenvalues of angular momentum we discussed previously, I obtain a differential equation for the function “P” which must be satisfied. You can see that this equation depends on both integers “l” and “m”, which is why I’ve included these integers as subscript labels for “P”. This equation is called the associated Legendre equation, and its solutions are called Legendre polynomials. And the resulting eigenfunctions, after combining these polynomials

for “theta” with our previous eigenfunction depending on “phi”, are known in mathematics as the spherical harmonic functions, which are very important in maths and physics. These spherical harmonic functions are the joint eigenfunctions of angular momentum.

[19] Let’s see a few examples of these spherical harmonic functions, and how they depend on “theta” and “phi”, for the first few values of “l” and “m”. Let’s consider the first three “l” values, and the potential values of “m” corresponding to each “l”. We recall that “m” can only range from “minus l” to “plus l”. Therefore for “l equals 0” we just have one spherical harmonic function, corresponding to “m equals 0”. The spherical harmonic then only consists of the normalisation constant, which we’ll discuss more in a moment. If I move to “l equals 1”, I now have three different possible values of “m”, and the three corresponding spherical harmonic functions are given here in the table. I can verify that each of these functions is a valid solution by substituting it in the associated Legendre equation. Next, let’s consider “l equals 2”, where I have five cases in the range from “m equals minus 2” to “m equals plus 2”. Each stage produces an increasingly complicated expression of sine and cosine functions!

[20] If I remember that these functions all depend on “theta” and “phi”, I can imagine that they have values which vary across a sphere. We can then visualise the spherical harmonic functions by plotting the magnitude of these values on a sphere. If I do so for the first few spherical harmonics, I can see that as “l” increases, there are a greater number of positive and negative fluctuations across the sphere, corresponding to these sine and cosine terms.

[21] The spherical harmonic functions possess the same properties as all eigenfunctions in quantum mechanics. First, these functions are orthogonal, which means when I multiply two different eigenfunctions together, taking the complex conjugate of the first, and integrate the product over all space, I will find zero. When I’m integrating over all space using spherical co-ordinates, I have to use the area element for spherical polar co-ordinates. This is why the “sine theta” term appears in this equation, since this term controls how much area is available at different heights of the sphere above and below the equator. If I multiply the same eigenfunctions together and integrate over the sphere, I obtain a normalisation relation, which is analogous to the need for the probability to integrate to “1” over all space. This condition can always be used to determine the normalisation constant associated with each spherical harmonic function. Finally, since any function can be expressed as a combination of eigenfunctions, it follows that any variation pattern across a sphere can be expressed as a sum of different spherical harmonic functions weighted by different coefficients. These coefficients can be determined using the orthogonality relation. So these are the key properties of spherical harmonic functions we need to know.

[22] Let’s now consider an example applying these ideas. Suppose we have a particle with a wavefunction over the sphere given by the form on the slide. This wavefunction describes the probability that the particle is located within different ranges of “theta” and “phi”. Of course a particle may also have some radial variation in three dimensions, but we’ll ignore that here and just look at the angular variation. We’ll ask, what values can be measured for the “z” component of angular momentum, and the total angular momentum? And what are the probabilities of finding each of these values? To solve this problem, we need to express this wavefunction as a sum over the relevant eigenfunctions, as we did in previous such

cases. In this case, those eigenfunctions are the spherical harmonic functions. Once we have expressed the wavefunction as a sum over the spherical harmonics, the square of the coefficients of each term will tell us the probability of finding each of the corresponding values in a measurement. To determine this expansion, I need to compare the terms in the wavefunction with the table of spherical harmonic functions from the previous slide. Doing this, I can adjust the coefficients of each spherical harmonic term to match the given wavefunction. I observe that if I include three different spherical harmonic functions – for "l equals 0, m equals 0", "l equals 1, m equals 1" and "l equals 1, m equals minus 1" – I can construct the total wavefunction. And the coefficients that I need to include with each spherical harmonic to do this are the square root of 3 divided by 4, and one divided by the square root of 8. You can verify that if I substitute the relevant spherical harmonic functions into this expression, I will recover the initial wavefunction. These coefficients can then be used to determine which angular momentum measurements are possible given the wavefunction, and with what probabilities. Starting with the total angular momentum, I can see there is one term with "l equals 0", and two terms with "l equals 1", and the probabilities of these cases are given by the squares of the coefficients. So the probability of measuring "l equals 0", which corresponds to a total angular momentum of zero, is three-quarters. And the probability of measuring "l equals 1", which corresponds to a total angular momentum of "2 times h-bar squared", is the sum of two one-eighths, or one-quarter. As a check, I can see that these probabilities sum to "1". Now thinking about the "z" component of angular momentum, this depends on the "m" value. There are three terms in the spherical harmonic expansion, which each have a different value of "m" equal to "minus 1", "0" and "plus 1". These terms correspond to angular momentum values of "minus h-bar", "zero" and "plus h-bar" respectively, and the respective probabilities can be deduced from the coefficients to be "one eighth", "three quarters" and "one eighth". Again, these probabilities sum to "1". We'll see further examples of this process during the activities!

[23] We'll end by summarising what we've learnt about angular momentum today. We started by deriving the operators for angular momentum using the classical formula, and demonstrating that whilst operators for the different components of angular momentum do not commute, the operator for the total angular momentum squared commutes with the operator for any single component. In this case, the total angular momentum and any one component may be simultaneously known, but I cannot know multiple components of angular momentum at the same time. Regarding angular momentum measurements, we saw that the eigenstates of angular momentum can be characterised by two integers or quantum numbers called "l" and "m", on which the eigenvalues depend. The eigenvalue for total angular momentum is given by "l times l plus 1 times h-bar squared", and the eigenvalue for the "z" component is given by "m times h-bar", where "m" is restricted to lie in a range from "minus l" to "plus l". Finally, we investigated the corresponding eigenfunctions. By writing the angular momentum operators in spherical polar co-ordinates, we associated these eigenfunctions with mathematical functions called spherical harmonics, which can form combinations to describe any function on the sphere. When those combinations are specified, the coefficients determine the angular momentum values which can be measured. We'll soon find that angular momentum is a very useful concept, as we move on to discuss solutions to the three-dimensional Schrödinger equation,

including the hydrogen atom. We'll discuss all this in the next video, and I'll see you soon for that!