Section 4: Angular Momentum

In these slides we will cover:

- Angular momentum operators in QM
- Commutation relations for these operators
- Implication for the simultaneous measurement of the total angular momentum and its components
- Ladder operators for angular momentum
- Eigenvalues of angular momentum
- Quantum states
- Eigenfunctions of angular momentum
- Measurement probabilities for angular momentum

Operators for angular momentum

What is angular momentum?

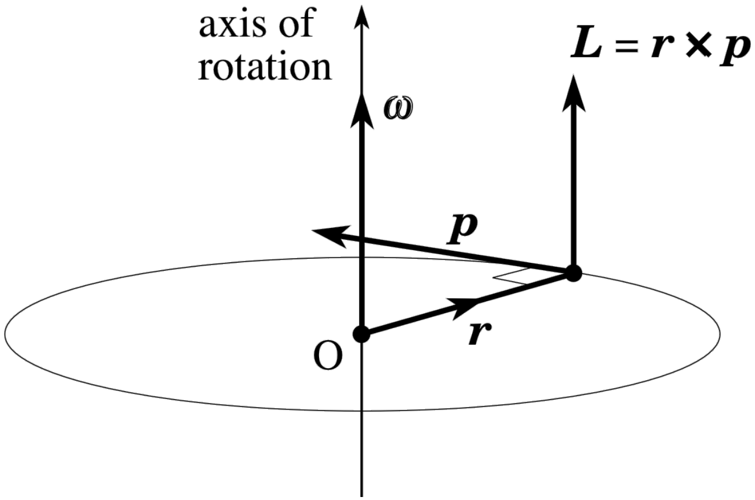
- Angular momentum is a confusing concept, so let’s recap!

Angular momentum is motion around an axis!

Operators for angular momentum

What is angular momentum?

- Angular momentum is a confusing concept, so let’s recap!
- The classical definition of the angular momentum $\vec{L}$ of a particle with momentum $\vec{p}$ at position $\vec{r}$ is, $\vec{L} = \vec{r} \times \vec{p}$



Angular momentum corresponds to the momentum multiplied by the perpendicular distance from an axis containing the origin, i.e. it describes motion relative to this axis

- Angular momentum is a **vector quantity** $\vec{L}$ since the axis of rotation is a **direction in space** (perpendicular to $\vec{r}$ and $\vec{p}$)

Operators for angular momentum

Angular momentum in quantum mechanics

- We already know that in Quantum Mechanics, physical observables are associated with corresponding operators

- We can deduce the **operators for the angular momentum components** $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ by classical analogy from $\hat{L} = \hat{r} \times \hat{p}$

- In 3D, the position and momentum operators are $\hat{r} = (\hat{x}, \hat{y}, \hat{z}) = (x, y, z)$ and $\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

- Hence,

$$\begin{pmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{pmatrix} = -i\hbar \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = -i\hbar \begin{pmatrix} y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{pmatrix}$$

Operators for angular momentum

Recapping commutation relations

- We recall from Section 2 that we can define **commutation relations** for pairs of operators: $\left[\hat{A}, \hat{B}\right] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}$
- The commutator $\left[\hat{A}, \hat{B}\right]$ is an operator itself, which is a combination of the other operators – it can be clearer to write this expression acting on a function: $\left[\hat{A}, \hat{B}\right] f = \hat{A}\,\hat{B} f - \hat{B}\,\hat{A} f$
- The expression $\hat{A}\,\hat{B} f$ means "apply $\hat{B}$ to the function f, then apply $\hat{A}$ to the resulting function"
- In Section 2 we discussed that if $\left[\hat{A}, \hat{B}\right] = 0$ then the two operators commute, have **simultaneous eigenfunctions**, and their observables can be **simultaneously known**

Operators for angular momentum

Commutation relations for angular momentum

- We can also deduce the operator for the **total angular momentum** $\hat{\boldsymbol{L}}$ by classical analogy: $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$
- We can use the definitions of the angular momentum operators to prove some important commutation relations they obey:

Base relation:

Equivalent versions from cyclic permutation of x, y, z:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \qquad [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

$$[\hat{L}_z, \hat{L}^2] = 0$$

$$[\hat{L}_x, \hat{L}^2] = 0 \qquad [\hat{L}_y, \hat{L}^2] = 0$$

Operators for angular momentum

Implication for simultaneous observables

- The commutation relation $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$ implies that **the components of angular momentum are not simultaneous observables** (i.e., only 1 component can be known at once)

- The commutation relation $[\hat{L}_z, \hat{L}^2] = 0$ implies that **the total angular momentum and any 1 component** (usually taken to be L_z for reasons we'll see shortly) **are simultaneous observables**

- In this case, we can come up with functions Y which are eigenfunctions of both $\hat{L}_z$ and $\hat{L}^2$, i.e. which satisfy:

$$\hat{L}^2\, Y = \lambda\, Y \qquad \hat{L}_z\, Y = \mu\, Y \qquad \text{Eigenvalues } \lambda, \mu$$

Angular momentum measurements

Ladder operators for angular momentum

- The next question is: what are these eigenvalues? We can find them using a **ladder operator technique**, similar to the one we recently employed for the harmonic oscillator. We define:

$$\hat{L}_+ = \hat{L}_x + i\,\hat{L}_y$$
$$\hat{L}_- = \hat{L}_x - i\,\hat{L}_y$$

- Here, $\hat{L}_x$ and $\hat{L}_y$ are the operators for the x- and y-components of angular momentum

- We can prove some useful relations:

$$\left[\hat{L}_z, \hat{L}_+\right] = \hbar\,\hat{L}_+ \qquad \left[\hat{L}^2, \hat{L}_+\right] = 0$$
$$\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\,\hat{L}_- \qquad \left[\hat{L}^2, \hat{L}_-\right] = 0$$

Angular momentum measurements

Ladder operators for angular momentum

- Non-examinable: Using these commutation relations, we can "wave a magic wand" as follows … *(you won't need to be able to replicate this slide and the next, I'm just including it in case you want to see the "small print" of how we can derive these angular momentum eigenvalues)*

- First: $\hat{L}^2(\hat{L}_\pm Y) = \hat{L}_\pm(\hat{L}^2 Y) = \hat{L}_\pm(\lambda Y) = \lambda(\hat{L}_\pm Y)$. If Y is an eigenfunction of $\hat{L}^2$, then so is $\hat{L}_\pm Y$, with the same eigenvalue

- Second: $\hat{L}_z(\hat{L}_\pm Y) = (\hat{L}_z\hat{L}_\pm Y - \hat{L}_\pm\hat{L}_z Y) + \hat{L}_\pm\hat{L}_z Y = \pm\hbar\hat{L}_\pm Y + \hat{L}_\pm(\mu\hbar) = (\mu \pm \hbar)(\hat{L}_\pm Y)$. If Y is an eigenfunction of $\hat{L}_z$ with eigenvalue μ, then so is $\hat{L}_\pm Y$, with eigenvalue $\mu \pm \hbar$

- **The operators $\hat{\boldsymbol{L}}_\pm$ move between different eigenfunctions of $\hat{\boldsymbol{L}}_z$, which have a fixed eigenvalue of $\hat{\boldsymbol{L}}^2$.** So far so good, we need one more piece…

Angular momentum measurements

Ladder operators for angular momentum

- Non-examinable: The final relation we can derive from the operators is,

$$\hat{L}^2 = \hat{L}_- \hat{L}_+ + \hat{L}_z^2 + \hbar \hat{L}_z$$

- Suppose $\mu = \hbar l$ is the eigenvalue of $\hat{L}_z$ corresponding to the highest value of L_z. This would mean:

$$\hat{L}_+ Y = 0 \qquad \hat{L}_z Y = \hbar l Y \qquad \hat{L}^2 Y = \lambda Y$$

- Using the formula at the top of the slide, $\hat{L}^2 Y = \hat{L}_- \hat{L}_+ Y + \hat{L}_z^2 Y + \hbar \hat{L}_z Y = 0 + \hbar^2 l^2 Y + \hbar^2 l Y = \hbar^2 l(l+1) Y$
- Hence, **the eigenvalue of $\boldsymbol{\hat{L}^2}$ is $\boldsymbol{l(l+1)\hbar^2}$**
- Further, the highest eigenvalue of L_z is $+\hbar l$, and from the previous slide they descend in integer steps of $\hbar$, all the way down to $L_z = -\hbar l$
- The **eigenvalues of $\boldsymbol{\hat{L}_z}$ are $\boldsymbol{m\hbar}$, where $\boldsymbol{m = -l, -l+1, \ldots, l-1, l}$**

Angular momentum measurements

Angular momentum eigenvalues

- What have we actually learned from all these manipulations??

1. The eigenvalues of $\hat{L}^2$ are $\boldsymbol{L^2 = l(l+1)\hbar^2}$ where l is a "quantum number" of total angular momentum

2. Given this value of l, the eigenvalues of $\hat{L}_z$ go from $-\hbar l$ to $+\hbar l$ in steps of $\hbar$

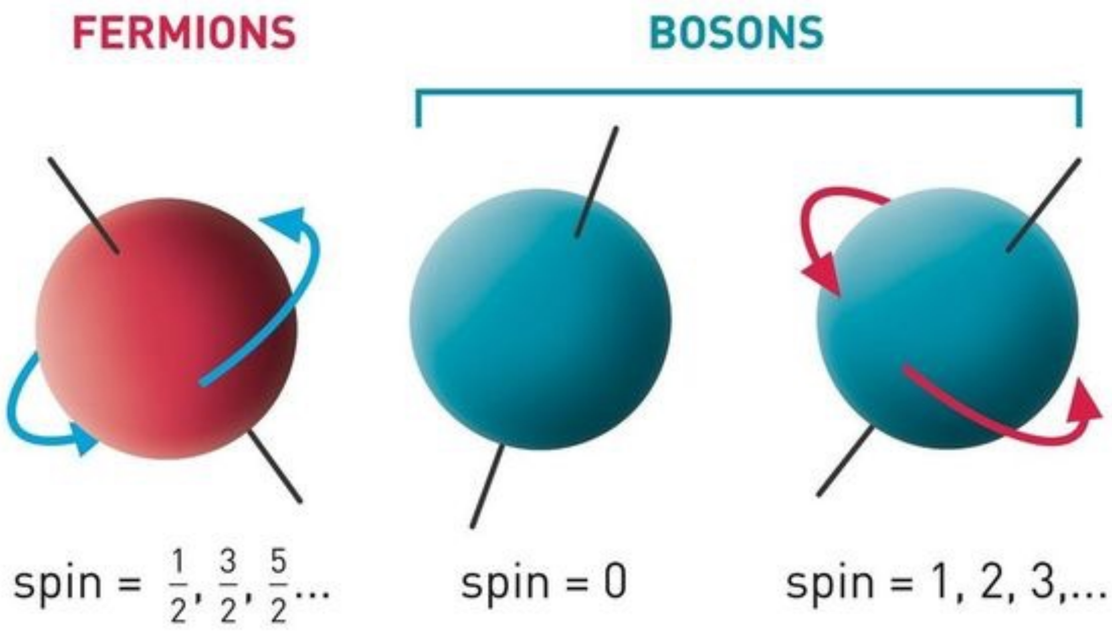
3. We can hence write these eigenvalues as $\boldsymbol{L_z = m\hbar}$, where $m = -l, -l+1, \dots, l-1, l$ where m is a quantum number of the z-component of angular momentum

4. For this to work, l has to be an **integer** (i.e. $0,1,2,\dots$) or **half-integer** (i.e. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$)

Angular momentum measurements

Angular momentum eigenvalues

- The final point on the previous slide tells us that particles can have either an **integer** or **half-integer** quantum number l of total angular momentum



Interesting aside: particles also have an **intrinsic** angular momentum, also known as **spin**. Particles with integer spin are known as **bosons**, and particles with half-integer spin are known as **fermions**. This determines some important properties!

Angular momentum measurements

Summary

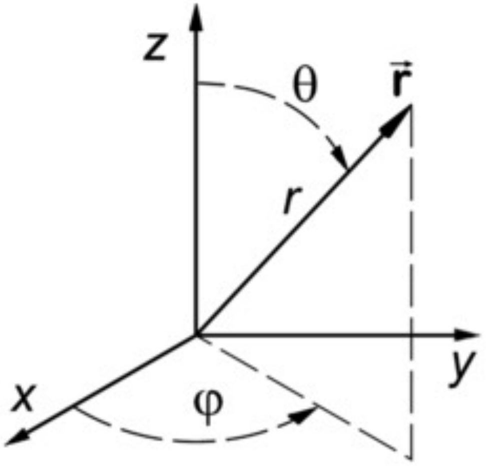
- The angular momentum eigenstates of a particle are characterised by two quantum numbers, l and m
- The quantum number l determines the total angular momentum, $L^2 = l(l+1)\hbar^2$
- The quantum number m determines the z-component of angular momentum, $L_z = m\hbar$
- For given l, m ranges from $-l$ to $+l$, separated by integers

Eigenfunctions of angular momentum

The eigenfunctions of $\hat{L}_z$

- Now we know the eigenvalues for angular momentum, let's determine the corresponding **eigenfunctions**
- Since angular momentum involves rotation, it's convenient to express the eigenfunctions in **spherical co-ordinates** (θ, ϕ)
- Recapping spherical polar co-ordinates:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$



Eigenfunctions of angular momentum

The eigenfunctions of $\hat{L}_z$

- We choose $\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ as the "base" component because it has a convenient expression in terms of spherical co-ordinates (θ, ϕ)

- We can derive this by considering the relation:

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

- Using this we immediately find:

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial \phi}$$

Just like linear momentum $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, but using an angular co-ordinate!

Eigenfunctions of angular momentum

The eigenfunctions of $\hat{L}_z$

- Using similar methods as before, the operator $\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$ has

Eigenfunctions $\frac{1}{\sqrt{2\pi}} e^{im\phi}$ Eigenvalues $m\hbar$ $m =$ integer

- The fact that m is an **integer** can be seen from the form of the eigenfunction: if I increase ϕ to $\phi + 2\pi$, I return to the exact same point in space, so the wavefunction must be unchanged

- The **normalisation** constant of the eigenfunction ensures that $\int_0^{2\pi} |\psi|^2 \, d\phi = 1$, where the limits are the allowed range of ϕ

Eigenfunctions of angular momentum

The joint eigenfunctions of $\hat{L}_z$ and $\hat{L}^2$

- Since the operators $\hat{L}_z$ and $\hat{L}^2$ commute, they have joint eigenfunctions which we write (suggestively!) as $Y_{lm}(\theta, \phi)$

- We already know that $e^{im\phi}$ are eigenfunctions of $\hat{L}_z$, so let's try writing for the eigenfunctions of $\hat{L}^2$:

$$Y_{lm}(\theta, \phi) = N\, P_{lm}(\theta)\, e^{im\phi}$$

- We now need the operator for $\hat{L}^2$! This is a bit messy (it is derived in textbooks), we'll just state it here:

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

Eigenfunctions of angular momentum

The joint eigenfunctions of $\hat{L}_z$ and $\hat{L}^2$

- Substituting in the expression $Y_{lm}(\theta,\phi) = P_{lm}(\theta)\, e^{im\phi}$ to the definition $\hat{L}^2\, Y_{lm}(\theta,\phi) = l(l+1)\hbar^2\, Y_{lm}(\theta,\phi)$ we find,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP_{lm}}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P_{lm} = 0$$

- This is called the **associated Legendre equation** and its solutions $P_{lm}(\theta)$ are called **Legendre polynomials**

- The resulting functions $Y_{lm}(\theta,\phi)$, which are joint eigenfunctions of $\hat{L}_z$ and $\hat{L}^2$, are known as the **spherical harmonic functions**

Eigenfunctions of angular momentum

Spherical harmonics

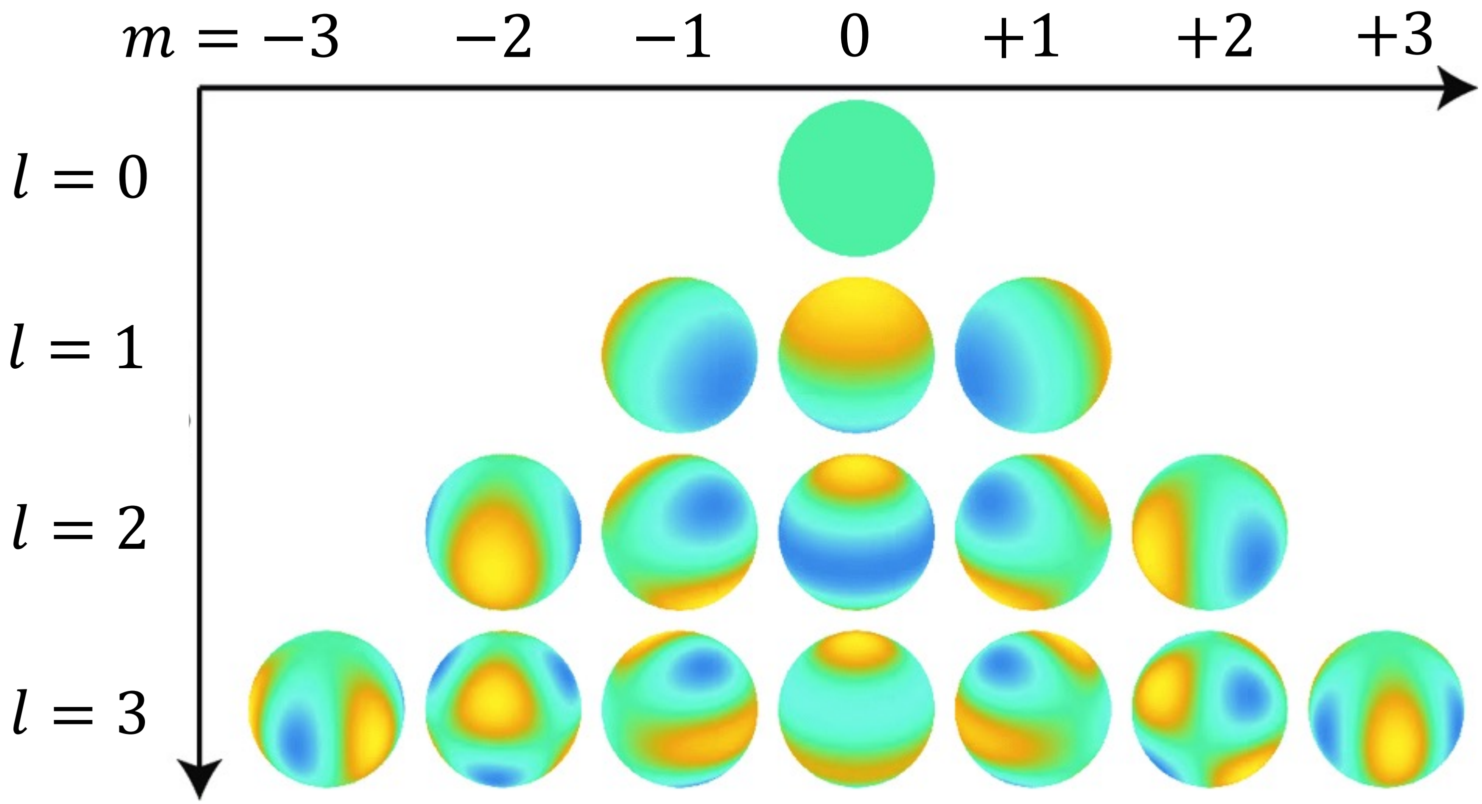
- Here are a few spherical harmonic functions $Y_{lm}(\theta, \phi)$ for the first few values of l and m:

	$l = 0$	$l = 1$	$l = 2$
$m = -2$			$Y_{2\,-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \; e^{-2i\phi}$
$m = -1$		$Y_{1\,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \; e^{-i\phi}$	$Y_{2\,-1} = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \; e^{-i\phi}$
$m = 0$	$Y_{00} = \sqrt{\frac{1}{4\pi}}$	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$
$m = +1$		$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \; e^{i\phi}$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \; e^{i\phi}$
$m = +2$			$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \; e^{2i\phi}$

Eigenfunctions of angular momentum

Spherical harmonics

- Here are some visualisations of $Y_{lm}(\theta, \phi)$ for some values of l and m (these all visualise the intensity on a sphere):



Eigenfunctions of angular momentum

Spherical harmonics

- The spherical harmonics have some key properties **which are the same as for any other set of eigenfunctions**:

- They are **orthogonal**:
$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = 0$$
if $l' \neq l$ and $m' \neq m$ *(note the area element for spherical co-ordinates)*

- They are **normalised**: $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, |Y_{lm}(\theta,\phi)|^2 = 1$

- Any function of (θ,ϕ) can be expressed as a **combination**:
$$\psi(\theta,\phi) = \sum_{lm} c_{lm} Y_{lm}(\theta,\phi)$$

- The coefficients: $c_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y^*_{lm}(\theta,\phi) \, \psi(\theta,\phi)$

Eigenfunctions of angular momentum

Example of angular momentum measurement

- A particle has wavefunction on a sphere, $\psi(\theta,\phi) = \sqrt{\frac{3}{16\pi}}(1+\sin\theta\cos\phi)$. What values of L_z and L^2 can be measured, with what probabilities?
- To solve this, express the wavefunction as a sum over eigenfunctions, $\psi(\theta,\phi) = \sum_{lm} c_{lm} Y_{lm}(\theta,\phi)$, then the probabilities are $|c_{lm}|^2$
- Using the formulae for Y_{lm}, we find $c_{00} = \sqrt{\frac{3}{4}}$ and $c_{11} = c_{1\,-1} = \frac{1}{\sqrt{8}}$:

$$\psi(\theta,\phi) = \sqrt{\frac{3}{4}} Y_{00}(\theta,\phi) + \frac{1}{\sqrt{8}} Y_{11}(\theta,\phi) - \frac{1}{\sqrt{8}} Y_{1\,-1}(\theta,\phi)$$

- Probability of $l = 0$ ($L^2 = 0$) is $\frac{3}{4}$, and probability of $l = 1$ ($L^2 = 2\hbar^2$) is $\frac{1}{4}$
- Probabilities of $m = -1,0,1$ ($L_z = -\hbar, 0, \hbar$) is $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$

Summary

Operators for angular momentum

- The angular momentum operators in Quantum Mechanics follow from $\hat{L} = \hat{r} \times \hat{p}$
- They obey commutation relations $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ and $[\hat{L}_z, \hat{L}^2] = 0$
- The total angular momentum and one component may be simultaneously known

Angular momentum measurements

- The eigenstates of angular momentum can be characterised by 2 quantum numbers l, m
- The eigenvalues are given by $l(l+1)\hbar^2$ (for L^2) and $m\hbar$ (for L_z) where l is an integer or half-integer, and $m = -l, -l+1, \dots, l-1, l$

Eigenfunctions of angular momentum

- The operator for the z-component of angular momentum can be represented as $\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$
- The eigenfunctions of angular momentum are the spherical harmonic functions, $Y_{lm}(\theta, \phi) = N\, P_{lm}(\theta)\, e^{im\phi}$