Quantum Mechanics Week 5: Class Prep solutions

Q1) The Bohr formula for the energy eigenvalues of the hydrogen atom is:

$$E_n = -\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2}$$

Q2) The 3D time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z) + V(x,y,z)\,\psi(x,y,z) = E\,\psi(x,y,z)$$

Q3) If $R(r) = u(r)/r$, consider:

$$r^2\frac{dR(r)}{dr} = r^2\frac{d}{dr}\left(\frac{u(r)}{r}\right) = r^2\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = r\frac{du(r)}{dr} - u(r)$$

Then:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r\frac{du(r)}{dr} - u(r)\right) = \frac{1}{r^2}\left(\frac{du}{dr} + r\frac{d^2u}{dr^2} - \frac{du}{dr}\right) = \frac{1}{r}\frac{d^2u(r)}{dr^2}$$

Q4) For the proposed solution $u_{10}(r) = re^{-r/a}$, we have:

$$\frac{du}{dr} = e^{-r/a} - \frac{r}{a}e^{-r/a}$$

$$\frac{d^2u}{dr^2} = -\frac{1}{a}e^{-r/a} - \frac{1}{a}e^{-r/a} + \frac{r}{a^2}e^{-r/a} = -\frac{2}{a}e^{-r/a} + \frac{r}{a^2}e^{-r/a}$$

Substituting in the radial equation for $l = 0$, this is a solution if:

$$-\frac{d^2u}{dr^2} - \frac{2}{ar}u = \frac{2\mu E}{\hbar^2}u$$

$$\rightarrow \frac{2}{a}e^{-r/a} - \frac{r}{a^2}e^{-r/a} - \frac{2}{ar}re^{-r/a} = \frac{2\mu E}{\hbar^2}re^{-r/a}$$

The first and third terms cancel out, so this is a solution with

$$E = -\frac{\hbar^2}{2\mu a^2} = -\frac{\hbar^2}{2\mu}\left(\frac{\mu e^2}{4\pi\varepsilon_0\hbar^2}\right)^2 = -\frac{\mu}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$$

which agrees with the Bohr formula for $n = 1$.

Q5) For the proposed solution $u_{21}(r) = r^2e^{-r/2a}$, we have:

$$\frac{du}{dr} = 2r\,e^{-r/2a} - \frac{r^2}{2a}e^{-r/2a} = \left(2r - \frac{r^2}{2a}\right)e^{-r/2a}$$

$$\frac{d^2u}{dr^2} = 2e^{-r/2a} - \frac{r}{a}e^{-r/2a} - \frac{r}{a}e^{-r/2a} + \frac{r^2}{4a^2}e^{-r/2a} = \left(2 - \frac{2r}{a} + \frac{r^2}{4a^2}\right)e^{-r/2a}$$

Substituting in the radial equation for $l = 1$, this is a solution if:

$$-\frac{d^2u}{dr^2} + \left(-\frac{2}{ar} + \frac{2}{r^2}\right)u = \frac{2\mu E}{\hbar^2}u$$

$$\rightarrow -\left(2 - \frac{2r}{a} + \frac{r^2}{4a^2}\right)e^{-r/2a} + \left(-\frac{2}{ar} + \frac{2}{r^2}\right)r^2e^{-r/2a} = \frac{2\mu E}{\hbar^2}r^2e^{-r/2a}$$

The first and fifth terms, and second and fourth terms, cancel out, so this is a solution with

$$E = -\frac{\hbar^2}{8\mu a^2} = -\frac{\mu}{8\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$$

which agrees with the Bohr formula for $n = 2$.

Q6) The number of distinct eigenfunctions with quantum number n is n^2, so there are 9 distinct eigenfunctions with $n = 3$.

Listing the possibilities, for $n = 3$ we can have $l = 0, 1, 2$:

- For $l = 0$ we can have $m = 0$ (1 state)
- For $l = 1$ we can have $m = -1, 0, 1$ (3 states)
- For $l = 2$ we can have $m = -2, -1, 0, 1, 2$ (5 states)

This makes a total of 9 possible states.