Quantum Mechanics Week 3: Class Prep solutions

Q1)

- $\psi(x)$ must be a continuous function of x
- $d\psi/dx$ must be a continuous function, except at an infinite discontinuity in $V(x)$

Q2) The time-independent Schrödinger equation for the harmonic oscillator is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\,\psi(x)=E\,\psi(x)$$

Q3) Using $\psi_1(x)=Ne^{-ax^2}$ we have,

$$\frac{d\psi_1}{dx}=-2ax\,Ne^{-ax^2}\qquad\frac{d^2\psi_1}{dx^2}=-2a\,Ne^{-ax^2}+4a^2x^2\,Ne^{-ax^2}$$

Substituting in the form of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left(-2a\,Ne^{-ax^2}+4a^2x^2\,Ne^{-ax^2}\right)+\frac{1}{2}m\omega^2x^2\left(Ne^{-ax^2}\right)=E\left(Ne^{-ax^2}\right)$$

Cancelling some terms,

$$-\frac{\hbar^2}{2m}(-2a+4a^2x^2)+\frac{1}{2}m\omega^2x^2\;=E$$

Using $a=m\omega/2\hbar$, the terms in x^2 cancel out, and the equation is satisfied if:

$$E=\frac{a\hbar^2}{m}=\frac{\hbar\omega}{2}$$

Q4) Applying the ladder operator $\hat{A}_+$ to $\psi_1(x)$:

$$\hat{A}_+\psi_1(x)=\left(\sqrt{a}\,x-\frac{1}{2\sqrt{a}}\frac{d}{dx}\right)Ne^{-ax^2}=\sqrt{a}\,x\,Ne^{-ax^2}-\frac{1}{2\sqrt{a}}\cdot-2axNe^{-ax^2}$$
$$=2\sqrt{a}\,x\,Ne^{-ax^2}\propto\;x\,e^{-ax^2}\propto\psi_2$$

Q5) Using $\psi_2(x)=Nxe^{-ax^2}$ we have,

$$\frac{d\psi_2}{dx}=Ne^{-ax^2}-2ax^2Ne^{-ax^2}$$
$$\frac{d^2\psi_2}{dx^2}=-2ax\,Ne^{-ax^2}-4ax\,Ne^{-ax^2}+4a^2x^3\,Ne^{-ax^2}=(-6a+4a^2x^2)\,Nxe^{-ax^2}$$

Substituting in the form of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}(-6a+4a^2x^2)\,Nxe^{-ax^2}+\frac{1}{2}m\omega^2x^2\left(Nxe^{-ax^2}\right)=E\left(Nxe^{-ax^2}\right)$$

Using $a=m\omega/2\hbar$, the terms in x^2 cancel out, and the equation is satisfied if:

$$E = \frac{3a\hbar^2}{m} = \frac{3\hbar\omega}{2}$$

Q6) The ladder operator $\hat{A}_+$ acts on an energy eigenfunction of the quantum harmonic oscillator to produce the next energy eigenfunction of higher energy.

Q7) The normalisation of the free-particle wavefunction represents the "intensity" of the beam, or the number of particles per unit length, or the average separation of particles in the beam.