Quantum Mechanics Week 1: Class Prep solutions

Q1) The time-dependent Schrödinger equation for the wavefunction is,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\,\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Q2) We'll substitute in the solution for $x > 0$, $\Psi(x,t) = Ne^{-\lambda x}e^{-i\omega t}$. Differentiating this wavefunction,

$$\frac{\partial\Psi}{\partial x} = -\lambda Ne^{-\lambda x}e^{-i\omega t} \rightarrow \frac{\partial^2\Psi}{\partial x^2} = \lambda^2 Ne^{-\lambda x}e^{-i\omega t} \qquad \frac{\partial\Psi}{\partial t} = -i\omega Ne^{-\lambda x}e^{-i\omega t}$$

Hence this wavefunction is a solution if,

$$-\frac{\hbar^2}{2m}\lambda^2 Ne^{-\lambda x}e^{-i\omega t} + V_0\,Ne^{-\lambda x}e^{-i\omega t} = i\hbar \cdot -i\omega Ne^{-\lambda x}e^{-i\omega t}$$
$$\rightarrow -\frac{\hbar^2}{2m}\lambda^2 + V_0 = \hbar\omega$$
$$\rightarrow \lambda = \sqrt{\frac{2m(V_0 - \hbar\omega)}{\hbar^2}}$$

The calculation for $x < 0$ is actually the same as above, so if we try this too we'll just get exactly the same result.

Q3) The modulus squared of the wavefunction is given by, for $x > 0$:

$$|\Psi(x,t)|^2 = \Psi^*\,\Psi = Ne^{-\lambda x}e^{+i\omega t} \cdot Ne^{-\lambda x}e^{-i\omega t} = N^2\,e^{-2\lambda x}$$

For $x < 0$ this is the same, just with a sign change in the exponential. Hence, this function is symmetrical about $x = 0$. Hence the normalisation condition is,

$$\int_{-\infty}^{\infty}|\Psi|^2\,dx = \int_{-\infty}^{0}|\Psi|^2\,dx + \int_{0}^{\infty}|\Psi|^2\,dx = N^2\int_{-\infty}^{0}e^{2\lambda x}\,dx + N^2\int_{0}^{\infty}e^{-2\lambda x}\,dx$$
$$= 2N^2\int_{0}^{\infty}e^{-2\lambda x}\,dx = 2N^2\left[\frac{e^{-2\lambda x}}{2\lambda}\right]_{\infty}^{0} = \frac{N^2}{\lambda} = 1$$

such that $N = \sqrt{\lambda}$.

Q4) The probability of the particle being located in the range $0 < x < 1/\lambda$ is,

$$\int_{0}^{1/\lambda}|\Psi|^2\,dx = N^2\int_{0}^{1/\lambda}e^{-2\lambda x}\,dx = \lambda\left[\frac{e^{-2\lambda x}}{2\lambda}\right]_{0}^{1/\lambda} = \frac{1}{2}\left(1 - \frac{1}{e^2}\right) = 0.43$$

Q5) The operator-eigenfunction-eigenvalue equation is,

$$\hat{A}\,\phi_n = a_n\,\phi_n$$

Q6) Applying the operator to the proposed eigenfunction we find,

$$\hat{A}\,\phi(x) = \frac{d}{dx}(e^{ax}) = a\,e^{ax} = a\,\phi(x)$$

Hence e^{ax} is an eigenfunction of the operator $\frac{d}{dx}$ with eigenvalue a.

Q7) The 3 key properties are,

- The eigenvalues of the operators are real (zero imaginary part)
- Different eigenfunctions of the operators are orthogonal
- Any function can be expressed as a linear combination of eigenfunctions