Quantum Mechanics Week 4: Class Activities solutions

Q1)

a) The equation with $l = 0$ and $m = 0$ is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{dP_{00}}{d\theta}\right) = 0$$

which is satisfied if $P_{00} = 1$, since $\frac{dP_{00}}{d\theta} = 0$.

b) The equation with $l = 1$ and $m = 0$ is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{dP_{10}}{d\theta}\right) + 2P_{10} = 0$$

Substituting in $P_{10} = \cos\theta$, we find:

$$\begin{aligned}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{d}{d\theta}\cos\theta\right) + 2\cos\theta &= \frac{1}{\sin\theta}\frac{d}{d\theta}(-\sin^2\theta) + 2\cos\theta \\ &= \frac{1}{\sin\theta}\ (-2\sin\theta\cos\theta) + 2\cos\theta \\ &= 0\end{aligned}$$

demonstrating that this form is a solution.

c) The equation with $l = 1$ and $m = 1$ is:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{dP_{11}}{d\theta}\right) + \left[2 - \frac{1}{\sin^2\theta}\right]P_{11} = 0$$

Substituting in $P_{11} = -\sin\theta$, we find:

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\ \frac{d}{d\theta}(-\sin\theta)\right) + \left[2 - \frac{1}{\sin^2\theta}\right](-\sin\theta) \\ &\qquad = -\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\cos\theta) - 2\sin\theta + \frac{1}{\sin\theta} \\ &= -\frac{1}{\sin\theta}(\cos^2\theta - \sin^2\theta) - 2\sin\theta + \frac{1}{\sin\theta} \\ &= -\frac{1}{\sin\theta}(1 - 2\sin^2\theta) - 2\sin\theta + \frac{1}{\sin\theta} \\ &= 0\end{aligned}$$

demonstrating that this form is a solution.

Hence, using the form $Y_{lm}(\theta,\phi) = N\ P_{lm}(\theta)\ e^{im\phi}$, the mathematical functions for Y_{00}, Y_{10} and Y_{11} are (ignoring the normalisation constant for now):

$$\begin{aligned}Y_{00} &\propto 1 \\ Y_{10} &\propto \cos\theta \\ Y_{11} &\propto -\sin\theta\ \ e^{i\phi}\end{aligned}$$

Q2)

The "$\sin\theta$" appears in the equation for normalising the spherical harmonic functions because it is part of the area element for integrations over a sphere. There is more area available near the equator than at the poles.

For Y_{00}:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \ N^2 = 1$$
$$\rightarrow 2\pi \cdot [\cos\theta]_{\pi}^{0} \cdot N^2 = 1$$
$$\rightarrow 2\pi \cdot 2 \cdot N^2 = 1$$
$$\rightarrow N = \frac{1}{\sqrt{4\pi}}$$

For Y_{10}:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \ N^2 \cos^2\theta = 1$$
$$\rightarrow 2\pi \cdot \left[\frac{\cos^3\theta}{3}\right]_{\pi}^{0} \cdot N^2 = 1$$
$$\rightarrow 2\pi \cdot \frac{2}{3} \cdot N^2 = 1$$
$$\rightarrow N = \sqrt{\frac{3}{4\pi}}$$

For Y_{11} (note the $e^{i\phi}$ cancels out in the complex conjugate):

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \ N^2 \sin^2\theta = 1$$

We can solve the integral of $\sin^3\theta$ over θ using the trig formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, hence $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$. Integrating,

$$\rightarrow 2\pi \cdot \left[\frac{3}{4}\cos\theta - \frac{1}{12}\cos 3\theta\right]_{\pi}^{0} \cdot N^2 = 1$$
$$\rightarrow 2\pi \cdot \frac{4}{3} \cdot N^2 = 1$$
$$\rightarrow N = \sqrt{\frac{3}{8\pi}}$$

Hence the normalised version of the wavefunctions are:

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta \ e^{i\phi}$$

Q3)

To demonstrate orthogonality we consider:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \ Y_{10}(\theta,\phi)\, Y_{11}^*(\theta,\phi)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \ \sin\theta \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)\left(-\sqrt{\frac{3}{8\pi}}\sin\theta \ e^{-i\phi}\right)$$

$$= \frac{3}{\sqrt{32}\pi}\left(\int_0^{2\pi} e^{-i\phi}\, d\phi\right)\left(\int_0^{\pi} d\theta \ \sin^2\theta \ \cos\theta\right)$$

$$= \frac{3}{\sqrt{32}\pi}\left[\frac{e^{-i\phi}}{-i}\right]_{2\pi}^{0}\left[\frac{\sin^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{3}{\sqrt{32}\pi}\left(\frac{1-1}{-i}\right)\left(\frac{0-0}{3}\right)$$

$$= 0$$

Q4) Using the spherical harmonic forms given at the end of Q2, we can express this wavefunction as a sum over Y_{00}, Y_{10} and Y_{11}:

$$\psi(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\left[\frac{1}{\sqrt{2}} + \cos\theta + \frac{1}{2}\sin\theta \, e^{i\phi}\right]$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{4\pi}} + \frac{1}{\sqrt{4\pi}}\cos\theta + \frac{1}{2}\frac{1}{\sqrt{4\pi}}\sin\theta \, e^{i\phi}$$

$$= \frac{1}{\sqrt{2}}Y_{00} + \frac{1}{\sqrt{3}}Y_{10} - \frac{1}{\sqrt{6}}Y_{11}$$

The modulus squared of each coefficient gives the probability associated with measuring corresponding eigenvalues. These probabilities are:

Term in expansion	(l, m)	Probability
Y_{00}	$l = 0, m = 0$	1/2
Y_{10}	$l = 1, m = 0$	1/3
Y_{11}	$l = 1, m = 1$	1/6

The possible results of measuring L^2 are:

- $l = 0$ or $L^2 = 0$, with probability $\frac{1}{2}$
- $l = 1$ or $L^2 = 2\hbar^2$, with probability $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

The possible results of measuring L_z are:

- $m = 0$ or $L_z = 0$, with probability $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
- $m = 1$ or $L_z = \hbar$, with probability $\frac{1}{6}$