Quantum Mechanics Week 3: Class Activities solutions

Q1) In the region $|x| < L$ the potential is $V(x) = 0$ and the Schrödinger equation takes the form:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

Trying the solution $\psi = A\cos kx$ we find:

$$-\frac{\hbar^2}{2m} \cdot -k^2 A\cos kx = E \cdot A\cos kx$$

which is hence a solution with $k^2 = 2mE/\hbar^2$.

Q2) In the region $|x| > L$ the potential is $V(x) = V_0$ and the Schrödinger equation takes the form:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

Trying the solution $\psi = Be^{\pm lx}$ we find:

$$-\frac{\hbar^2}{2m} \cdot l^2 Be^{\pm lx} + V_0 \cdot Be^{\pm lx} = E \cdot Be^{\pm lx}$$

which is hence a solution with $l^2 = 2m(V_0 - E)/\hbar^2$.

Q3) The continuity of $\psi(x)$ at $x = L$ implies:

$$A\cos kL = Be^{-lL}$$

where we have taken the decaying form of the exponential solution to ensure $\psi \to 0$ as $x \to \infty$, such that the wavefunction can be normalised. We note that the continuity of $\psi(x)$ at $x = -L$ results in the same equation.

The continuity of $\frac{d\psi}{dx}$ at $x = L$ implies:

$$-kA\sin kL = -lBe^{-lL}$$

Again, the continuity of $\frac{d\psi}{dx}$ at $x = -L$ results in the same equation.

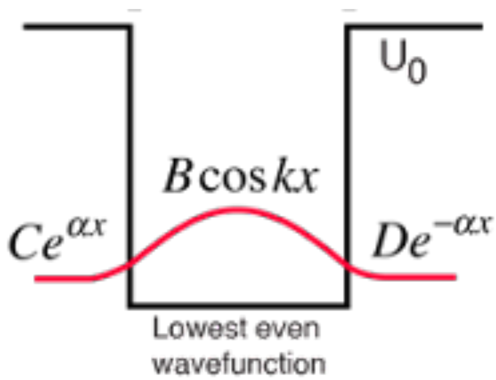
Dividing these two equations we find:

$$\tan kL = \frac{l}{k}$$

Now using the expressions for k and l from above:

$$\tan\sqrt{\frac{2mEL^2}{\hbar^2}} = \sqrt{\frac{V_0}{E} - 1}$$

Q4) The wavefunction corresponding to the lowest energy state will consist of a cosine solution in the central region with the longest possible wavelength, and decaying exponentials on either side (image credit from http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/pfbox.html):



Q5) The continuity of $\psi(x)$ at $x = 0$ implies:

$$\Psi_I(0,t) + \Psi_R(0,t) = \Psi_B(0,t)$$
$$\rightarrow e^{-i\omega t} + Re^{-i\omega t} = (A + B)\, e^{-i\omega t}$$
$$\rightarrow 1 + R = A + B \qquad [\text{Eq. }1]$$

The continuity of $\psi(x)$ at $x = L$ implies:

$$\Psi_B(x,t) = \Psi_T(x,t)$$
$$\rightarrow (Ae^{kL} + Be^{-kL})\, e^{-i\omega t} = Te^{i(kL-\omega t)}$$
$$\rightarrow Te^{ikL} = Ae^{kL} + Be^{-kL} \qquad [\text{Eq. }2]$$

where we have used $l = k$. The continuity of $\frac{\partial\psi}{\partial x}$ at $x = 0$ implies:

$$\frac{\partial\Psi_I(0,t)}{\partial x} + \frac{\partial\Psi_R(0,t)}{\partial x} = \frac{\partial\Psi_B(0,t)}{\partial x}$$
$$\rightarrow ike^{-i\omega t} - ik\, Re^{-i\omega t} = (kA - kB)\, e^{-i\omega t}$$
$$\rightarrow (1 - R)i = A - B \qquad [\text{Eq. }3]$$

The continuity of $\frac{\partial\psi}{\partial x}$ at $x = L$ implies:

$$\frac{\partial\Psi_B(0,t)}{\partial x} = \frac{\partial\Psi_T(0,t)}{\partial x}$$
$$\rightarrow l(Ae^{kL} - Be^{-kL})\, e^{-i\omega t} = ikTe^{i(kL-\omega t)}$$
$$\rightarrow i\, Te^{ikL} = Ae^{kL} - Be^{-kL} \qquad [\text{Eq. }4]$$

Q6) We can now find the unknown coefficients from Eq.1-4. First we consider:

$$[\text{Eq. }2] + [\text{Eq. }4] \rightarrow (1 + i)\, Te^{ikL} = 2Ae^{kL} \qquad [\text{Eq. }5]$$
$$[\text{Eq. }2] - [\text{Eq. }4] \rightarrow (1 - i)\, Te^{ikL} = 2Be^{-kL} \qquad [\text{Eq. }6]$$

Dividing [Eq.5] by [Eq.6] gives:

$$\frac{B}{A} = \left(\frac{1-i}{1+i}\right) e^{2kL} \qquad [\text{Eq. } 7]$$

Re-arranging [Eq.5] gives:

$$\frac{T}{A} = \frac{2\, e^{kL}\, e^{-ikL}}{1+i} \qquad [\text{Eq. } 8]$$

Now we determine A using the other 2 equations:

$$\begin{aligned}
[\text{Eq. } 1] - [\text{Eq. } 3] \times i &\rightarrow 2 = (1-i)A + (1+i)B \\
&\rightarrow 2 = (1-i)A + (1+i)\left(\frac{1-i}{1+i}\right) e^{2kL} A \\
&\rightarrow 2 = A(1-i)(1+e^{2kL}) \\
&\rightarrow A = \frac{2}{(1-i)(1+e^{2kL})}
\end{aligned}$$

Now we find T using [Eq.8]:

$$T = \frac{2\, e^{kL}\, e^{-ikL}}{1+i} \cdot \frac{2}{(1-i)(1+e^{2kL})} = \frac{2\, e^{-ikL}}{e^{kL} + e^{-kL}}$$

which leads to:

$$|T|^2 = \frac{4}{(e^{kL} + e^{-kL})^2}$$