# Quantum Mechanics Week 2: Class Activities solutions

Q1) The eigenfunctions of momentum, with eigenvalue $p$, have the form:

$$\phi(x) = \frac{1}{\sqrt{2L}} e^{ipx/\hbar}$$

which have been normalised within the range of the infinite potential well, $-L < x < L$. To find which momentum values can be measured given a wavefunction, we must express the wavefunction as a linear combination of eigenfunctions. This can be done by transforming the sine and cosine functions into complex exponentials using:

$$\sin x = (e^{ix} - e^{-ix})/2i$$
$$\cos x = (e^{ix} + e^{-ix})/2$$

With this substitution, the wavefunction becomes:

$$\begin{aligned}\psi(x) &= \frac{2}{\sqrt{5L}} \sin\left(\frac{\pi x}{L}\right)\left[1 + \cos\left(\frac{\pi x}{L}\right)\right] \\ &= \frac{2}{\sqrt{5L}}\left[\frac{e^{i\pi x/L} - e^{-i\pi x/L}}{2i}\right]\left[1 + \frac{e^{i\pi x/L} + e^{-i\pi x/L}}{2}\right] \\ &= \frac{1}{2i\sqrt{5L}}\left(-e^{-2i\pi x/L} - 2e^{-i\pi x/L} + 2e^{i\pi x/L} + e^{2i\pi x/L}\right)\end{aligned}$$

This expression is now a sum over the momentum eigenfunctions $\phi_p(x) = \frac{1}{\sqrt{2L}} e^{ipx/\hbar}$, where we can read off the momentum values by comparing the exponents:

$$\psi(x) = -\frac{1}{i\sqrt{10}}\phi_{p=-2\hbar\pi/L} - \frac{2}{i\sqrt{10}}\phi_{p=-\hbar\pi/L} + \frac{2}{i\sqrt{10}}\phi_{p=\hbar\pi/L} + \frac{1}{i\sqrt{10}}\phi_{p=2\hbar\pi/L}$$

Hence the possible momentum values are $p = \left(-\frac{2\hbar\pi}{L}, -\frac{\hbar\pi}{L}, \frac{\hbar\pi}{L}, \frac{2\hbar\pi}{L}\right)$ which have probabilities given by the modulus squared of the coefficients, which are $\left(\frac{1}{10}, \frac{2}{5}, \frac{2}{5}, \frac{1}{10}\right)$, which as a check, we can see sum to 1.

These answers can also be deduced immediately from the expression,

$$\psi \propto \left(-e^{-2i\pi x/L} - 2e^{-i\pi x/L} + 2e^{i\pi x/L} + e^{2i\pi x/L}\right)$$

without requiring the normalisation. That's because we can already see the momentum values in this expression (as the exponential coefficients), and the relative probabilities are the squares of the amplitudes which are (1,4,4,1). We can then normalise these probabilities so they sum to 1, after which they become $\left(\frac{1}{10}, \frac{2}{5}, \frac{2}{5}, \frac{1}{10}\right)$.

Q2) The expectation value of momentum is given by a weighted sum over the eigenvalues using the corresponding probabilities:

$$\langle p \rangle = \sum_n |c_n|^2 \, p_n = \frac{1}{10} \cdot -\frac{2\hbar\pi}{L} + \frac{2}{5} \cdot -\frac{\hbar\pi}{L} + \frac{2}{5} \cdot \frac{\hbar\pi}{L} + \frac{1}{10} \cdot \frac{2\hbar\pi}{L} = 0$$

The expectation value of a momentum measurement is zero.

Q3) In Week 1, we showed that this wavefunction could be written as a sum of two energy eigenfunctions $\phi_n(x)$:

$$\Psi(x,0) = \frac{2}{\sqrt{5L}} \sin\left(\frac{\pi x}{L}\right)\left[1 + \cos\left(\frac{\pi x}{L}\right)\right] = \frac{2}{\sqrt{5}}\phi_2(x) + \frac{1}{\sqrt{5}}\phi_4(x)$$

Following the recipe for time evolution, we attach a complex exponential $e^{-iEt/\hbar}$ to each eigenfunction:

$$\Psi(x,t) = \frac{2}{\sqrt{5}}\phi_2(x)\, e^{-iE_2 t/\hbar} + \frac{1}{\sqrt{5}}\phi_4(x)\, e^{-iE_4 t/\hbar}$$

where $E_2 = \frac{\pi^2\hbar^2}{2mL^2}$ and $E_4 = \frac{2\pi^2\hbar^2}{mL^2}$.

Q4) The formula for the expectation value is:

$$\langle E\rangle = \int_{-\infty}^{\infty} \Psi^* \widehat{H} \Psi\, dx$$

Substituting in the wavefunction from Q3,

$$\langle E\rangle = \int_{-\infty}^{\infty} \left[\frac{2}{\sqrt{5}}\phi_2^*(x)\, e^{iE_2 t/\hbar} + \frac{1}{\sqrt{5}}\phi_4^*(x)\, e^{iE_4 t/\hbar}\right] \widehat{H} \left[\frac{2}{\sqrt{5}}\phi_2(x)\, e^{-iE_2 t/\hbar} + \frac{1}{\sqrt{5}}\phi_4(x)\, e^{-iE_4 t/\hbar}\right] dx$$

We now apply the operator $\widehat{H}$ to the eigenfunctions in the final bracket, using the result $\widehat{H}\phi_n(x) = E_n\phi_n(x)$, and multiply out the expression:

$$\langle E\rangle = \frac{4}{5}E_2 \int_{-\infty}^{\infty} |\phi_2(x)|^2\, dx + \frac{2}{5}E_4\, e^{i(E_2-E_4)t/\hbar} \int_{-\infty}^{\infty} \phi_2^*\, \phi_4\, dx + \frac{2}{5}E_2\, e^{-i(E_2-E_4)t/\hbar} \int_{-\infty}^{\infty} \phi_4^*\, \phi_2\, dx + \frac{1}{5}E_4 \int_{-\infty}^{\infty} |\phi_4(x)|^2\, dx$$

Using the fact that the eigenfunctions are orthogonal and normalised, the second and the third terms are zero and this expression simplifies to:

$$\langle E\rangle = \frac{4}{5}E_2 + \frac{1}{5}E_4 = \frac{4}{5}\cdot\frac{\pi^2\hbar^2}{2mL^2} + \frac{1}{5}\cdot\frac{2\pi^2\hbar^2}{mL^2} = \frac{4\pi^2\hbar^2}{5mL^2}$$

which is constant in time.

Q5) The expectation value of position squared is:

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\, |\psi(x)|^2\, dx = \frac{1}{L}\int_{-L}^{L} x^2 \cos^2\left(\frac{\pi x}{2L}\right) dx$$

With the substitution, $u = \pi x/2L$, hence $du = \pi\, dx/2L$:

$$\langle x^2\rangle = \frac{1}{L}\left(\frac{2L}{\pi}\right)^3 \int_{-\pi/2}^{\pi/2} u^2 \cos^2 u\, du = \frac{16L^2}{\pi^3}\int_0^{\pi/2} u^2 \cos^2 u\, du$$

where in the final expression we have used the fact that the integral is symmetric to change the lower limit to zero. Using the given result,

$$\langle x^2 \rangle = \frac{16L^2}{\pi^3}\left[\frac{1}{6}\cdot\frac{\pi^3}{8} - \frac{1}{4}\cdot\frac{\pi}{2}\right] = L^2\left(\frac{1}{3} - \frac{2}{\pi^2}\right) = 0.131\ L^2$$

The expectation value of momentum squared is:

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p}\hat{p}\psi\ dx = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{d^2\psi}{dx^2}\ dx = -\frac{\hbar^2}{L}\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right)\cdot -\left(\frac{\pi}{2L}\right)^2 \cos\left(\frac{\pi x}{2L}\right)\ dx$$

$$= \frac{\hbar^2\pi^2}{4L^3}\int_{-L}^{L} \cos^2\left(\frac{\pi x}{2L}\right)\ dx = \frac{\hbar^2\pi^2}{4L^3}\cdot\frac{1}{2}\cdot 2L = \frac{\hbar^2\pi^2}{4L^2} = 9.87\ \frac{\hbar^2}{4L^2}$$

The square root of the product of these is:

$$\sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \sqrt{0.131\ L^2 \cdot 9.87\ \frac{\hbar^2}{4L^2}} = \sqrt{1.29 \cdot \frac{\hbar^2}{4}} = 1.14\ \frac{\hbar}{2}$$

This is hence consistent with the uncertainty principle requirement: $\sqrt{\langle x^2 \rangle \langle p^2 \rangle} \geq \frac{\hbar}{2}$.