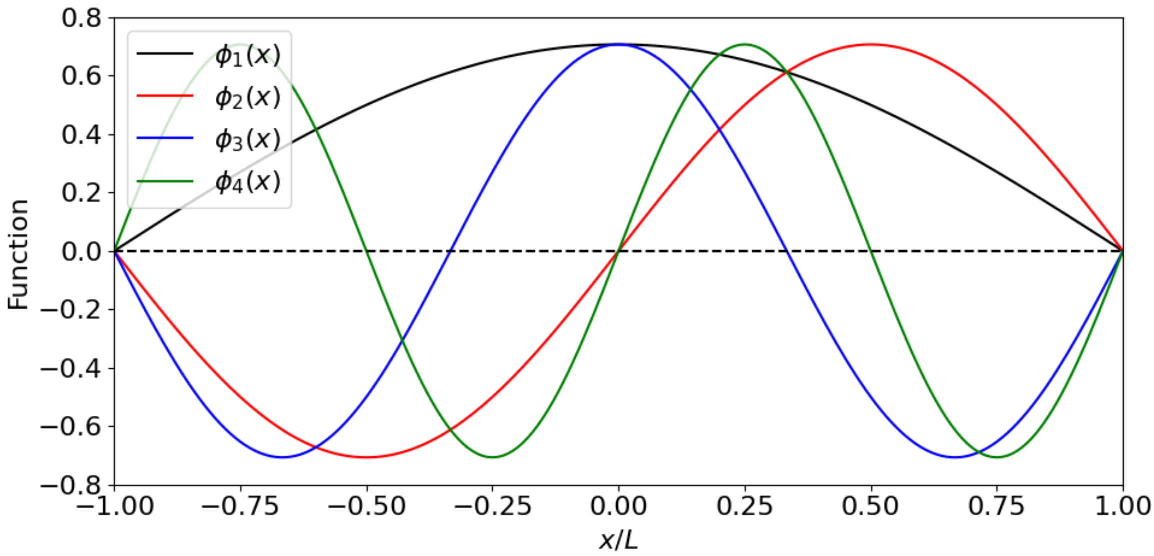
Quantum Mechanics Week 1: Class Activities solutions

Q1) The plot looks like:



Q2) We'll use the trig formula: $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\int_{-\infty}^{\infty} |\psi_1|^2 \, dx = \frac{1}{L}\int_{-L}^{L} \cos^2\left(\frac{\pi x}{2L}\right) dx = \frac{1}{2L}\int_{-L}^{L}\left[1 + \cos\left(\frac{\pi x}{L}\right)\right] dx = \frac{1}{2L}\left[x + \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\right]_{-L}^{L}$$

$$= \frac{1}{2L}[(L + 0) - (-L + 0)] = \frac{1}{2L}\cdot 2L = 1$$

$$\int_{-\infty}^{\infty} \psi_1^* \, \psi_2 \, dx = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right)\sin\left(\frac{\pi x}{L}\right) dx = 0$$

In the second integral, we have used the fact that this is an integral of the product of an even and odd function over a symmetric interval. A different solution method would use the trig formula $\sin 2\theta = 2\sin\theta\cos\theta$.

Q3) We can arrange this wavefunction as a sum of sine/cosine terms by multiplying out the bracket and using $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\psi(x) = \frac{2}{\sqrt{5L}}\sin\left(\frac{\pi x}{L}\right)\left[1 + \cos\left(\frac{\pi x}{L}\right)\right]$$

$$= \frac{2}{\sqrt{5L}}\sin\left(\frac{\pi x}{L}\right) + \frac{2}{\sqrt{5L}}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)$$

$$= \frac{2}{\sqrt{5L}}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{5L}}\sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{2}{\sqrt{5}}\phi_2(x) + \frac{1}{\sqrt{5}}\phi_4(x)$$

where in the last line we have used,

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{2L}\right), & n = 1,3,5,\ldots \\ \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{2L}\right), & n = 2,4,6,\ldots \end{cases}$$

Q4) We use the principle that when the wavefunction is expanded as a sum over eigenfunctions, $\psi(x) = \sum_n c_n\, \phi_n(x)$, the results of the measurement are the corresponding eigenvalues, with probabilities $|c_n|^2$.

Here, the energy eigenvalues are given by $E_n = \frac{n^2\pi^2\hbar^2}{8mL^2}$.

Hence we can measure $E_2 = \frac{\pi^2\hbar^2}{2mL^2}$ with probability $\left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}$, and $E_4 = \frac{2\pi^2\hbar^2}{mL^2}$ with probability $\left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5}$.

Q5) The easiest approach is to use the weighted sum of eigenvalues:

$$\langle E \rangle = \sum_n |c_n|^2\, E_n = \frac{4}{5}E_2 + \frac{1}{5}E_4 = \frac{4}{5} \cdot \frac{\pi^2\hbar^2}{2mL^2} + \frac{1}{5} \cdot \frac{2\pi^2\hbar^2}{mL^2} = \frac{4\pi^2\hbar^2}{5mL^2}$$

Q6) Consider the expression,

$$\int_{-\infty}^{\infty} |\psi|^2\, dx = \int_{-\infty}^{\infty} \psi^*\psi\, dx = \int_{-\infty}^{\infty} \left(\sum_m c_m^* \phi_m^*\right)\left(\sum_n c_n \phi_n\right) dx$$
$$= \sum_m \sum_n c_m^* c_n \int_{-\infty}^{\infty} \phi_m^* \phi_n\, dx = \sum_n |c_n|^2$$

This shows that if the wavefunction is normalised, then $\sum_n |c_n|^2 = 1$.

Q7) The expression for the expectation value is,

$$\langle a \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi\, dx = \int_{-\infty}^{\infty} \left(\sum_m c_m^* \phi_m^*\right) \hat{A} \left(\sum_n c_n \phi_n\right) dx$$
$$= \int_{-\infty}^{\infty} \left(\sum_m c_m^* \phi_m^*\right)\left(\sum_n c_n a_n \phi_n\right) dx = \sum_m \sum_n c_m^* c_n a_n \int_{-\infty}^{\infty} \phi_m^* \phi_n\, dx$$
$$= \sum_n |c_n|^2 a_n$$