General Relativity Problem Set 5 (Classes 9-11)

Black holes

Q1) Imagine you are at rest at $r = 16GM/c^2$, near a black hole whose mass M is a million solar masses. You launch a clock radially outward at such a speed that it comes to rest at $r = 32GM/c^2$. The clock subsequently falls radially back into the black hole.

a) How much time does the launched clock register between its launch and the time that it comes to rest?

b) How much time does it register between rest and crossing the event horizon?

c) How much time does it register between crossing the event horizon and its destruction at the origin?

You may start from the equations in Class 9: $\frac{dt}{d\tau} = \frac{K}{A}$ and $\frac{1}{c}\frac{dr}{d\tau} = \sqrt{K^2 - A}$, where $A = 1 - \frac{R_S}{r}$.

Note that the integral $\int \sqrt{\frac{16x}{16-x}}\, dx$ may be solved by the substitution $x = 16\sin^2\theta$, producing the result $128\int \sin^2\theta \, d\theta = 64\left[\theta - \frac{1}{2}\sin 2\theta\right]$.

Cosmology

Q2) If the only kind of matter in the Universe were ordinary baryonic matter, then the matter density parameter would be $\Omega_m \approx 0.04$, which is very small. An approximation to such a Universe would be a completely empty Universe model where $\Omega_m = \Omega_\Lambda = 0$. Assume that the Hubble parameter $H_0 = 70\ km\ s^{-1}\ Mpc^{-1}$.

a) Use the Friedmann equation to show the scale factor $a(t) = H_0\, t$ in this Universe.

b) Write down the metric for this Universe.

c) Find the age of this model Universe, and compare it with 13.2 Gyr, which is our best estimate of the age of the oldest star in our Galaxy.

d) The Universe became transparent when $z = 1090$. How many years was this after the Big Bang, in this model Universe?

e) Imagine that we observe a galaxy with $z = 1$. How long ago did light leave the galaxy? What is the co-ordinate distance of the galaxy?

Note that the integral $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x$.