General Relativity Problem Set 4 (Classes 7-9)

Riemann tensor

Q1) Consider the following metric for a two-dimensional space with co-ordinates (p, q):

$$ds^2 = dp^2 + e^{2ap}\, dq^2$$

where a is a constant. Find the non-zero Christoffel symbols and the only non-zero component of the Riemann tensor $R^{\kappa}_{\lambda\mu\nu}$. Does this metric describe a flat or curved space?

The Schwarzschild metric

Q2) Hubble Space Telescope measurements of Sirius' white dwarf companion show that the white dwarf has a mass of $1.02\ M_{\odot}$ and a radius of 5640 km (with 2% uncertainties), and that spectral features of its light have a fractional redshift of 2.68×10^{-4} (with 6% uncertainty). Are these results consistent with General Relativity?

Q3) In Christopher Nolan's 2014 science fiction epic *Interstellar*, Matthew McConaughey plays an astronaut whose spacecraft visits a planet located near a black hole named "Gargantua", which has a mass $M = 10^8\ M_{\odot}$. Without giving away too many spoilers, a central plot point is that on this planet, 1 hour corresponds to 7 years in an observer's frame far from the black hole.

a) Assuming the Schwarzschild metric, determine the radial co-ordinate of the planet's orbit that would produce this amount of gravitational time dilation.

b) List three problems that would make this scenario physically impossible!

The film (with comparatively impressive scientific accuracy!) evades these problems by specifying that Gargantua is rotating with angular momentum J. In that case, it is incorrect to assume the Schwarzschild metric and we should instead assume that the spacetime geometry is described by the more complicated Kerr metric:

$$ds^2 = -\left(1 - \frac{R_s r}{\Sigma}\right) c^2 dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \left(r^2 + a^2 + \frac{R_s r a^2}{\Sigma} \sin^2\theta\right) \sin^2\theta\, d\phi^2 - \frac{2R_s r a \sin^2\theta}{\Sigma}\, c\, dt\, d\phi$$

where $R_s = 2GM/c^2$ is the Schwarzschild radius, $a = J/Mc$, $\Sigma = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 - R_s r + a^2$.

c) Use this metric to find a relationship between r, M, J and θ for which 1 hour of proper time $d\tau$ would correspond to 7 years of co-ordinate time dt.

d) Assume the planet is in the innermost stable orbit of the Kerr metric, which has a radial co-ordinate $r = R_s/2$. If $\theta = \pi$, what is the value of J?