# General Relativity Problem Set 3 (Classes 5-7)

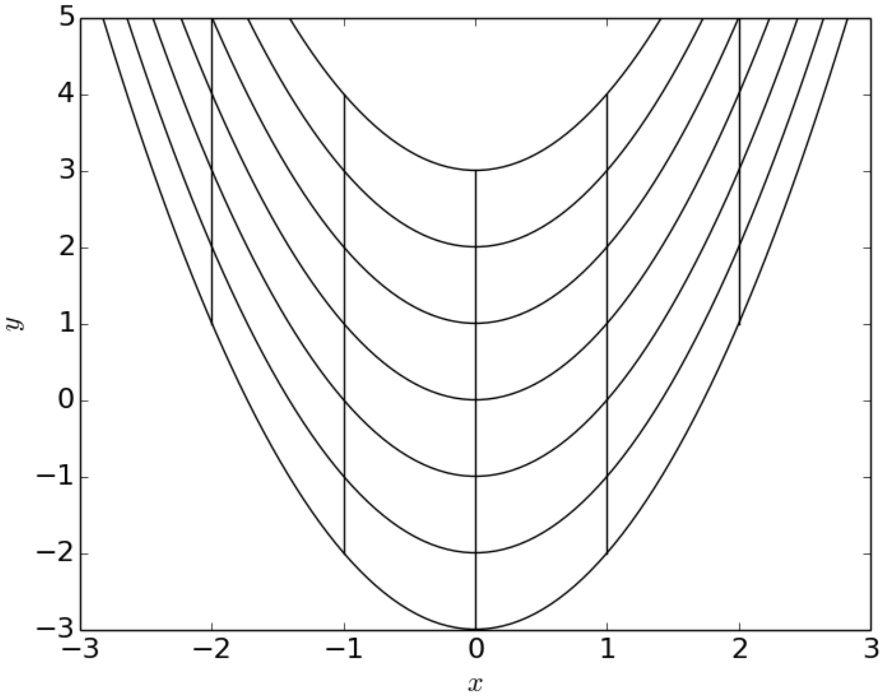
## Metrics and Geodesics

In this week's homework we will practise using metrics in different co-ordinate systems, and solving for the geodesics in a given metric. To keep things (relatively!) simple, we will restrict ourselves to 2 dimensions.

Consider a parabolic co-ordinate system in 2D space, $(p, q)$. The transformation functions from ordinary Cartesian $(x, y)$ co-ordinates to these parabolic co-ordinates are:

$$p = x$$
$$q = y - x^2$$

The $(p, q)$ co-ordinate system is just another way – admittedly, quite a strange way! – to create a co-ordinate grid in 2 dimensions. Here is a plot of the $(p, q)$ co-ordinate grid – i.e. lines of constant $p$ and $q$ – in the Cartesian $(x, y)$ frame:




a) The metric in the Cartesian space is $ds^2 = dx^2 + dy^2$, or $g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Use the transformation $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$ to show that the metric for the $(p, q)$ co-ordinates is:

$$g'_{\mu\nu} = \begin{pmatrix} 1 + 4p^2 & 2p \\ 2p & 1 \end{pmatrix}$$

b) A vector $\vec{A}$ has components in $(p,q)$ co-ordinates, $A^p = 1, A^q = 0$. What are its components in $(x,y)$ co-ordinates?

c) Show that the length of the vector $\vec{A}$ in the previous part – that is, $A_\mu A^\mu$ – has the same value in both co-ordinate systems.

Let's now determine an equation for the geodesics – that is, the lines of shortest separation between two points – in $(p,q)$ co-ordinates. Geodesics $x^\mu(s)$ in any co-ordinate system always satisfy the equation,

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \left(\partial_\lambda g_{\mu\kappa} - \frac{1}{2}\partial_\mu g_{\kappa\lambda}\right)\frac{dx^\kappa}{ds}\frac{dx^\lambda}{ds} = 0$$

d) Show that the $\mu = p$ equation becomes:

$$(1+4p^2)\frac{d^2p}{ds^2} + 2p\frac{d^2q}{ds^2} + 4p\left(\frac{dp}{ds}\right)^2 = 0$$

and the $\mu = q$ equation becomes:

$$2p\frac{d^2p}{ds^2} + \frac{d^2q}{ds^2} + 2\left(\frac{dp}{ds}\right)^2 = 0$$

Hence, by combining the above two equations, show that the geodesics must satisfy

$$\frac{d^2p}{ds^2} = 0$$

e) Use the result for part d) to derive the general form of the geodesics, $p(s)$ and $q(s)$.

f) Use the result for part e) to obtain the geodesics in Cartesian $(x,y)$ co-ordinates, and explain why this allows you to check your solution.

g) If $x^\mu(s)$ is a geodesic, then the Christoffel symbols are defined by:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{ds}\frac{dx^\lambda}{ds} = 0$$

By comparison of this equation and the formulae in part d), write down any non-zero Christoffel symbols for the $(p,q)$ co-ordinates.