General Relativity Problem Set 2 (Classes 3-5)

Electromagnetism

Q1) If $F^{\mu\nu}$ is the Maxwell electromagnetic field tensor,

a) Show that $\eta_{\mu\nu}F^{\mu\nu} = 0$.

b) Show that $\partial_\mu \partial_\nu F^{\mu\nu} = 0$.

c) $F_{\mu\nu}F^{\mu\nu}$ is a scalar quantity which has the same value in all inertial reference frames – what is its value in terms of the amplitudes of the electric and magnetic fields?

Q2) In this problem we will derive the electric field of a charge $+q$ moving with constant velocity v through a frame S. The charge is at rest in frame S'.

a) Write down the electric and magnetic fields in S'.

b) Use the Lorentz transformations to show the electric field in S is $\vec{E} = \left(E'_x, \gamma E'_y, \gamma E'_z\right)$.

c) At time $t = 0$, the frames coincide. At this time, show that $\vec{r}' = (\gamma x, y, z)$.

d) Hence show that the electric field in S, at $t = 0$, is given by the expression

$$\vec{E} = \frac{q\vec{r}}{4\pi\varepsilon_0 \gamma^2 r^3 [1 - (v \sin\theta / c)^2]^{3/2}}$$

where θ is the angle between $\vec{r}$ and the x-axis.

Accelerated Motion

Q3) An object is moving in the $+x$ direction in a certain inertial frame with a speed

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(at+1)^2}}$$

where a is a constant, and $t \geq 0$ is the time in that frame.

a) Find the t-component of the 4-velocity of the object, u^t, as a function of t.

b) Show that the proper time τ elapsed along the path of the particle is given by $a\tau = \ln(at+1)$.

c) Express u^t as a function of τ.

d) Express the x-component of the 4-velocity of the particle, u^x, as a function of both t and τ.

e) Find expressions for $t(\tau)$ and $x(\tau)$.