General Relativity Problem Set 1 (Classes 1-3)

Special Relativity

Q1) Here are 3 short problems which review Special Relativity.

a) Two spacecraft, A and B, travel between space stations Alpha and Beta, which are at rest with respect to each other in deep space 6 Tm apart (1 Tm $= 10^{12}$ m). Alpha and Beta together comprise an inertial reference frame. Both spacecraft leave station Alpha at noon, as read by clocks on Alpha, and both arrive at Beta 12 hours later, as read by clocks on Beta (whose clocks are synchronized with those on Alpha). Spacecraft A travels straight from Alpha to Beta at a constant velocity, while spacecraft B travels at a constant speed on a semi-circular path of radius 3 Tm. Calculate the trip time as measured by clocks on spacecraft A, and as measured by clocks on spacecraft B.

b) Imagine that a train is moving at a speed of $0.8\ c$. A passenger points a laser out of the train window perpendicular to the tracks, and the laser emits a brief flash of light. Use the transformation of 4-velocity to determine the angle the velocity of this light flash makes with the tracks, in the ground frame.

c) A positive pion π^+ (with rest mass 140 MeV/c^2) at rest will decay after an average lifetime of 26 ns to an anti-muon μ^+ (with rest mass 106 MeV/c^2) and a muon neutrino ν_μ with negligible mass. Use conservation of 4-momentum to determine the outgoing neutrino's energy. (Hint: treat the neutrino as if it were a photon).

Index Notation

Q2) Which are the following are validly-constructed index equations? (Consider only the equation's structure, ignore its meaning). If they are not valid, what is the problem?

a) $0 = m^2 + (p^\mu)^2$

b) $\frac{dF^{\mu\nu}}{d\tau} = 0$

c) $\frac{dp^\mu}{d\tau} = g$ (g is a constant $\neq 0$)

d) $F_{\alpha\beta} = \eta_{\alpha\mu}\, \eta_{\beta\nu}\, F^{\mu\sigma}$

e) $A^{\alpha\beta} = \eta_{\alpha\mu}\, \eta_{\beta\nu}\, F^{\mu\nu}$ (A is a matrix)

f) $A^\mu = \delta^\mu{}_\alpha\, A^\alpha$ (A^μ is a 4-vector)

g) $0 = A^\mu + B^\nu$ (A^μ and B^ν are 4-vectors)

h) $q\, F^{\mu\nu} = \frac{dp^\mu}{d\tau}$

Q3) In which of the following cases have I renamed indices in a valid way? (Consider only the equation's structure, ignore its meaning). If they are not valid, what is the problem?

a) $A^2 = \eta_{\alpha\beta}\, A^\alpha\, A^\beta \quad \rightarrow \quad A^2 = \eta_{\mu\nu}\, A^\alpha\, A^\beta$

b) $0 = \eta_{\alpha\beta}\, A^\beta + \eta_{\alpha\mu}\, B^\mu \quad \rightarrow \quad 0 = \eta_{\alpha\beta}\left(A^\beta + B^\beta\right)$

c) $\eta_{\mu\nu} = \eta_{\alpha\beta}\, L^\alpha{}_\mu\, L^\beta{}_\nu \quad \rightarrow \quad \eta_{\mu\nu} = \eta_{\alpha\alpha}\, L^\alpha{}_\mu\, L^\alpha{}_\nu$

d) $\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu}\, \eta_{\nu\alpha}\, u^\alpha \quad \rightarrow \quad \frac{dp^\mu}{d\tau} = q\, F^{\mu\nu}\, \eta_{\nu\mu}\, u^\mu$

e) $(L^{-1})^\alpha{}_\mu\, \eta_{\alpha\nu} = \eta_{\mu\beta}\, L^\beta{}_\nu \quad \rightarrow \quad (L^{-1})^\beta{}_\mu\, \eta_{\beta\nu} = \eta_{\mu\alpha}\, L^\alpha{}_\nu$