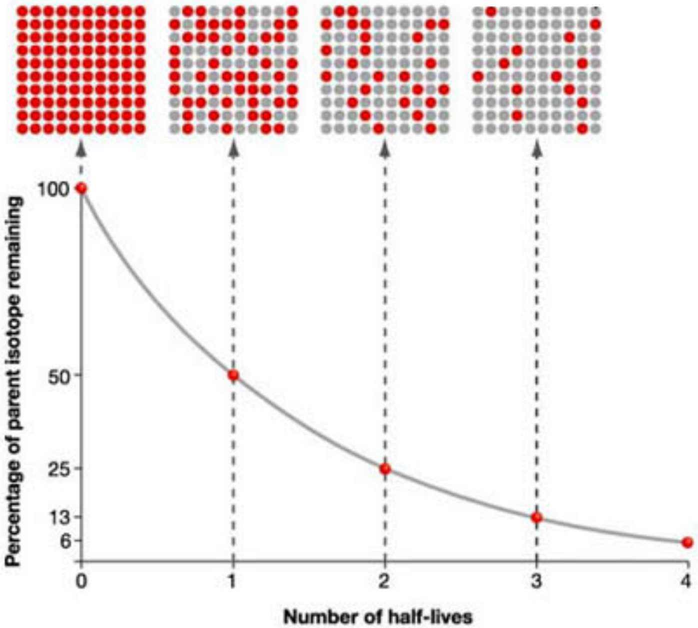
PHY20004 Nuclear Physics Class 4:
Radioactive Decay Law

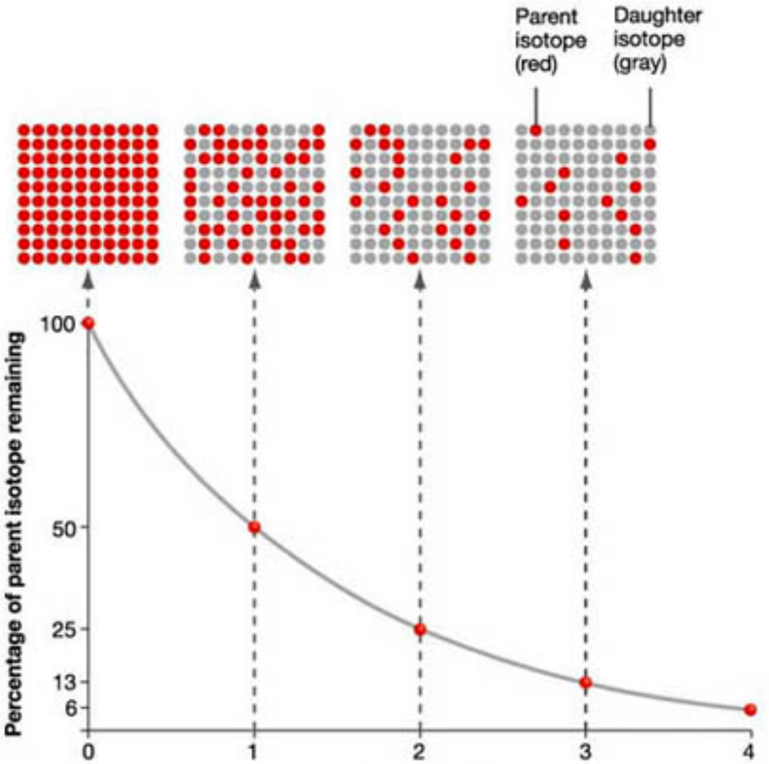
In this class we will study the mathematical law which describes the statistics of radioactive decays with time, and its powerful applications



The statistics of radioactivity

- Radioactive decay is a random process governed by probability
- We **cannot** predict when an individual nucleus decays, but we **can** predict the average number of decays of a large number

We cannot predict the result of a single dice roll, but we can predict the number of 6's when a large number of dice are rolled



Credit: http://ch302.cm.utexas.edu

Mathematics of decay

- The probability that an individual nucleus will decay in a time interval dt is $dP = \lambda\, dt$, where λ is a parameter called the **decay constant**

- Therefore, given a large number of nuclei N, the number of nuclei that decay in time dt is $dN_{\mathrm{decay}} = N\, \lambda\, dt$

- This **reduces** the number, so $dN = -dN_{\mathrm{decay}} = -N\, \lambda\, dt$

The decay rate is hence given by:

$$\frac{dN}{dt} = -\lambda\, N$$

Solving this equation for the number remaining at time t:

$$N(t) = N(0)\, e^{-\lambda t}$$

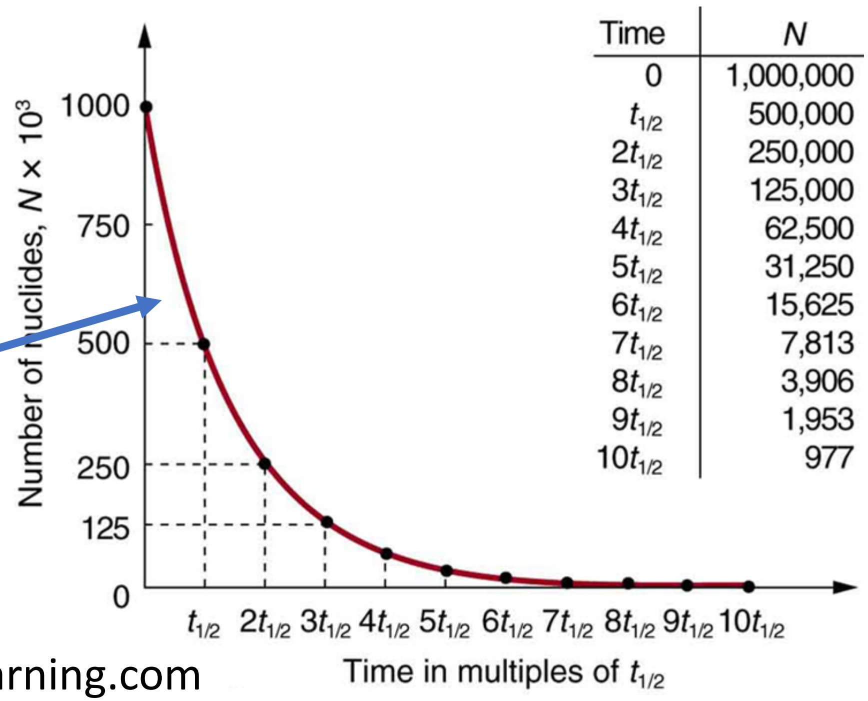
- This is the famous **law of exponential decay**

Mathematics of decay

- The **radioactive half-life** $t_{1/2}$ is the time required for *half the nuclei in a given sample to decay*

$$N(t_{1/2}) = \frac{1}{2}N(0) \rightarrow \frac{1}{2}N(0) = N(0)\,e^{-\lambda t_{1/2}} \rightarrow t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

Population drops by $\frac{1}{2}$ in each time interval $t_{1/2}$



Time	N
0	1,000,000
$t_{1/2}$	500,000
$2t_{1/2}$	250,000
$3t_{1/2}$	125,000
$4t_{1/2}$	62,500
$5t_{1/2}$	31,250
$6t_{1/2}$	15,625
$7t_{1/2}$	7,813
$8t_{1/2}$	3,906
$9t_{1/2}$	1,953
$10t_{1/2}$	977

Credit: https://courses.lumenlearning.com

Mathematics of decay

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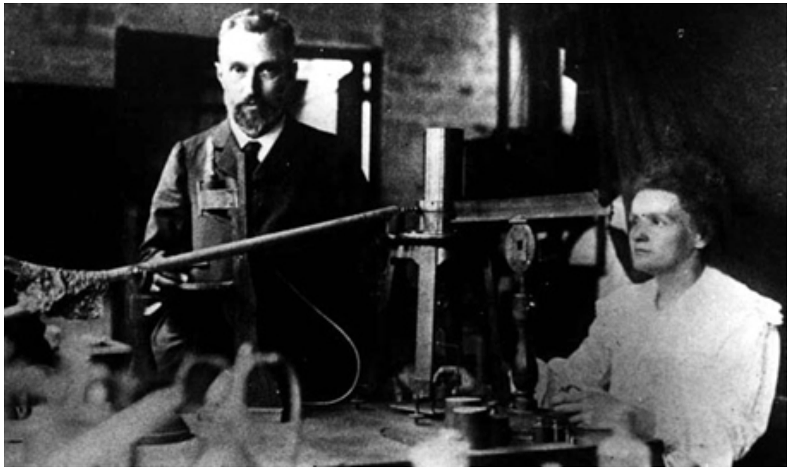
- The **mean lifetime** $\bar{t}$ is the average time taken for a nucleus to decay, given probability of surviving to time t is $\frac{N(t)}{N(0)} = e^{-\lambda t}$

$$\bar{t} = \frac{\int_0^\infty t\, P(t)\, dt}{\int_0^\infty P(t)\, dt} = \frac{\int_0^\infty t\, e^{-\lambda t}\, dt}{\int_0^\infty e^{-\lambda t}\, dt} \longrightarrow \bar{t} = \frac{1}{\lambda}$$

Units of radioactivity

- The level of radioactivity of a sample is measured as the **number of decays per second** ($\frac{dN_{\text{decay}}}{dt} = \lambda N$)
- 1 **Becquerel** (Bq) sample radioactivity is equivalent to 1 decay per second (this is a very small number)
- A more practical unit is the **Curie** (Ci), defined such that $1\ \text{Ci} = 3.7 \times 10^{10}\ \text{Bq}$

Marie and Pierre Curie
(Nobel Prizes 1903, 1911)

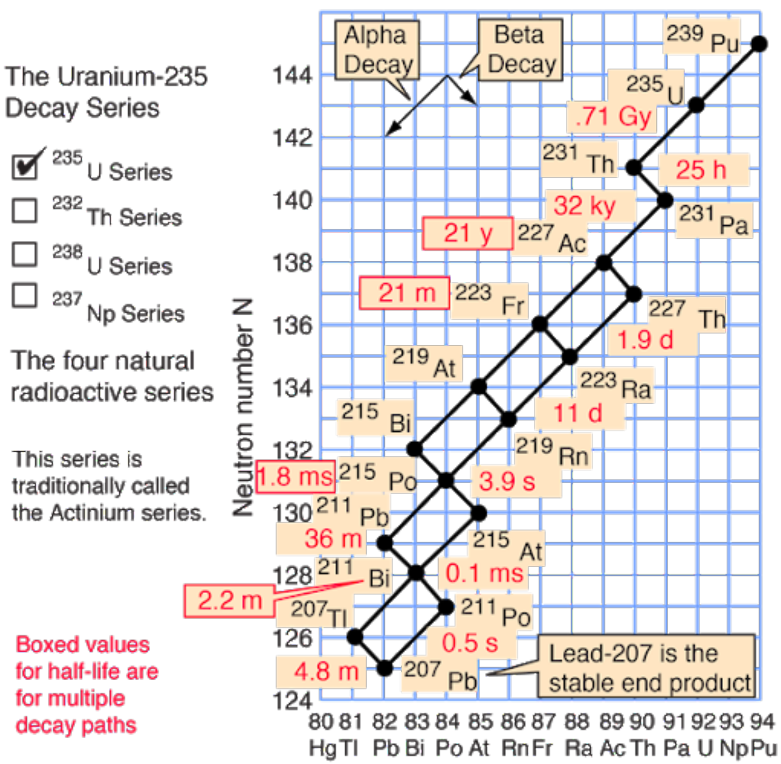


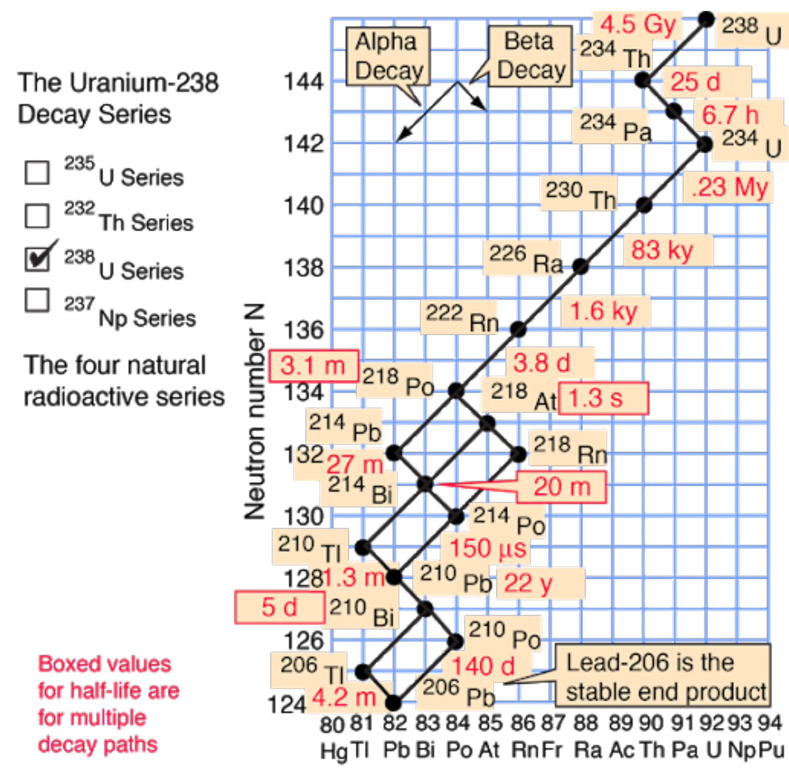
Radioactive decay chains

- The products of radioactive decay may themselves be unstable to decay – creating a **decay chain**

α-decay creates a nucleus with $(Z-2, N-2)$

β-decay creates a nucleus with $(Z+1, N-1)$





Credit: http://hyperphysics.phy-astr.gsu.edu/

Radioactive decay chains

- To model the population of a daughter nuclide forming from a parent (with decay constant λ_1), whilst itself decaying (with decay constant λ_2), we need to use coupled equations …

Modelling a decay chain

Let $N_1(t)$ and $N_2(t)$ be the number of parent and daughter nuclei at time t, respectively. We'll assume $N_1(0) = N_0$ and $N_2(0) = 0$. From the decay laws:

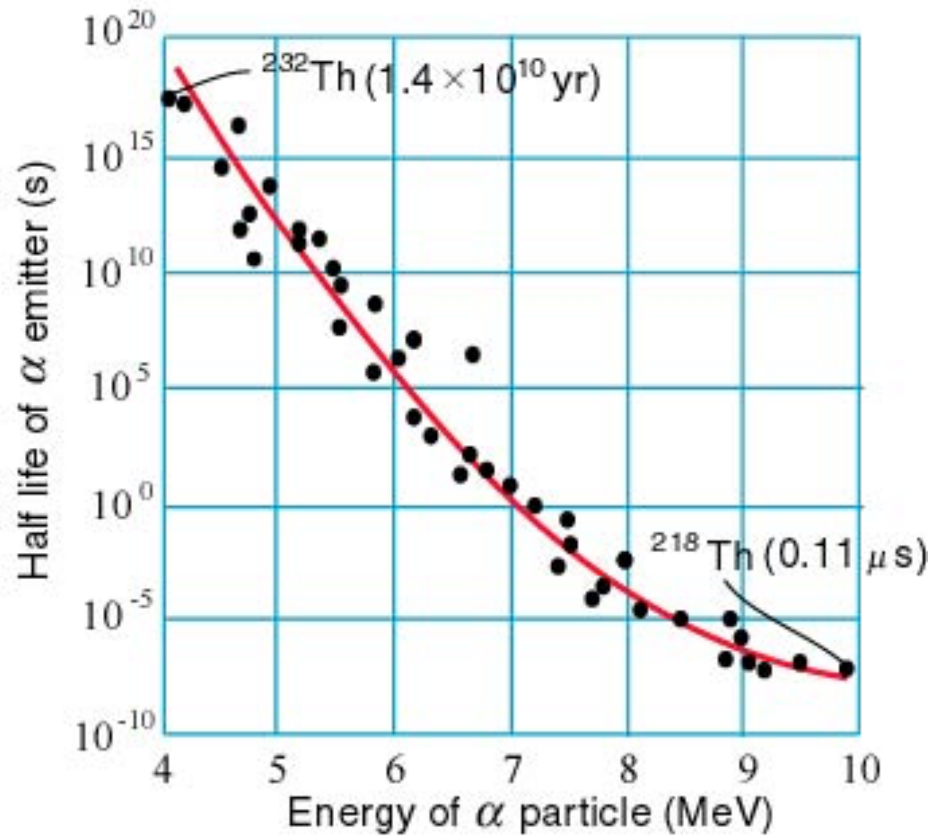
$$dN_1 = -\lambda_1 \, N_1(t) \, dt$$
$$dN_2 = +\lambda_1 \, N_1(t) \, dt - \lambda_2 \, N_2(t) \, dt$$

Solving the 1st equation: $N_1(t) = N_0 \, e^{-\lambda_1 t}$

Try a solution to the 2nd equation: $N_2(t) = A \, e^{-\lambda_1 t} + B \, e^{-\lambda_2 t}$. Using the condition $N_2(0) = 0$, we find $A + B = 0$. Substituting in, we find $N_2(t) = \frac{\lambda_1 \, N_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$

More on half-lives

- The **half-life** of nuclides varies dramatically, from a fraction of a second to the age of the Universe!

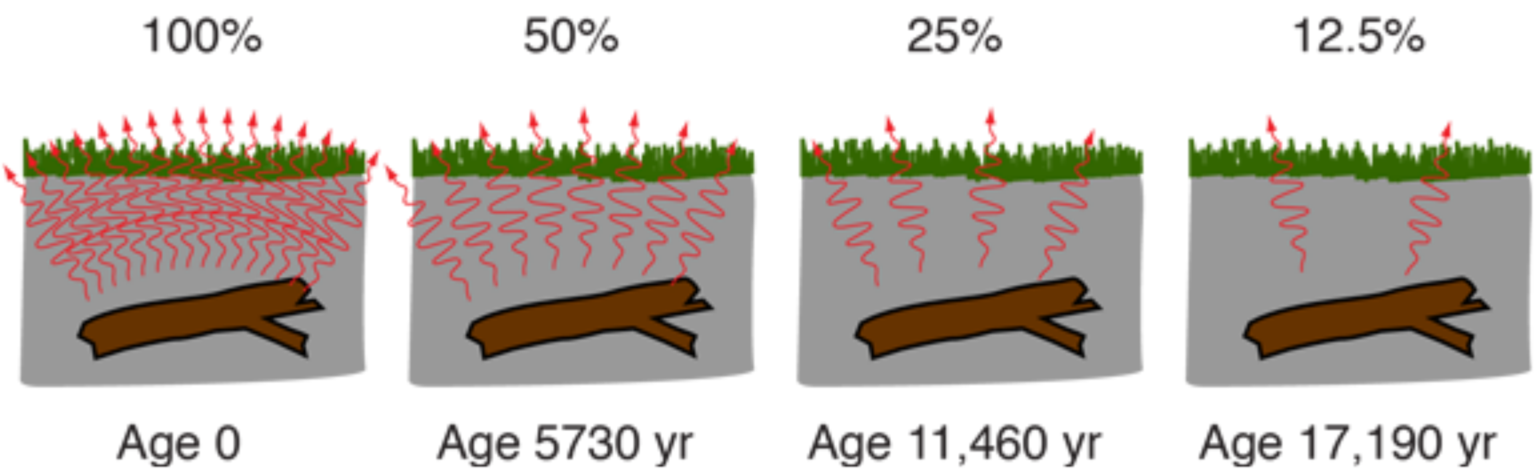


Credit: http://ne.phys.kyushu-u.ac.jp

- Example from α-decay – ranges over 30 orders of magnitude!
- This is because α-decay depends on quantum tunneling, which is exponentially sensitive to the height of the energy barrier
- Different half-lives can be used in different applications

Carbon dating

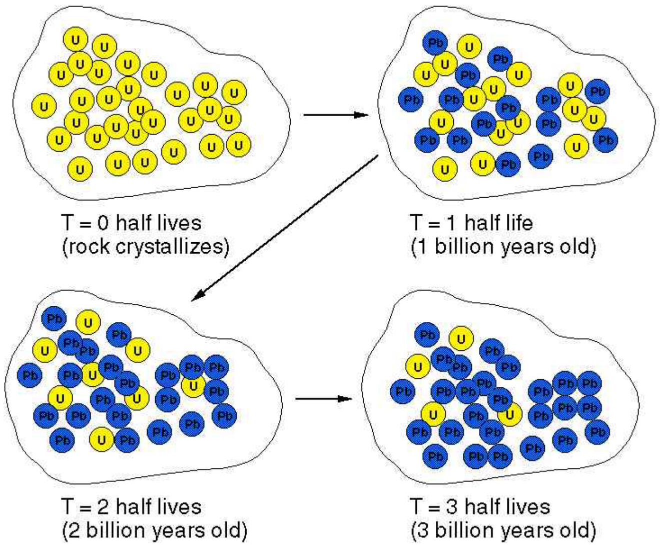
- The isotope $^{14}\mathrm{C}$ is continuously produced in the atmosphere by cosmic ray bombardment, mixing with the dominant isotope $^{12}\mathrm{C}$
- Living organisms continually exchange carbon with the atmosphere, and hence have a fixed isotopic abundance ratio $^{14}\mathrm{C}/^{12}\mathrm{C} \approx 1.3\times10^{-12}$
- After death, $^{14}\mathrm{C}$ β-decays into $^{14}\mathrm{N}$ with a half-life of $5{,}730$ yr, whereas $^{12}\mathrm{C}$ is stable. Hence the $^{14}\mathrm{C}/^{12}\mathrm{C}$ ratio declines, **tracing the object's age**

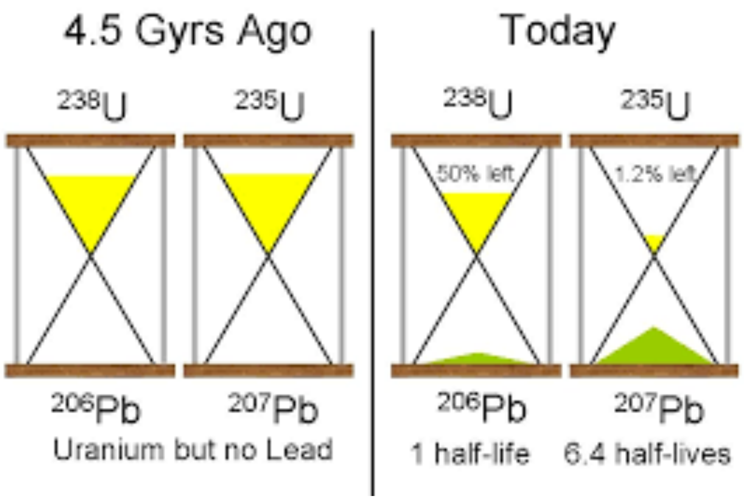


Credit: http://hyperphysics.phy-astr.gsu.edu/hbase/Nuclear/cardat.html

Geological dating

- Radioactive dating using nuclides with longer half-lives can be used to **estimate the age of rocks**, or even the Earth itself
- There are several versions, but in general we need to compare the relative abundance of **parent** and **daughter** nuclides
- **Uranium-lead** dating uses the decay chains of $^{235}\text{U} \rightarrow {}^{207}\text{Pb}$ (half-life 7.1×10^{8} yr) and $^{238}\text{U} \rightarrow {}^{206}\text{Pb}$ (half-life 4.47×10^{9} yr)





Credit: http://geologylearn.blogspot.com, http://www.astronomy.ohio-state.edu

Key take-aways

- Radioactive decay is a **spontaneous, random process** governed by the laws of probability
- A radioactive population of nuclei declines as $N(t) = N(0)\, e^{-\lambda t}$ with **decay constant** λ
- A population has **radioactive half-life** $t_{1/2} = 0.693/\lambda$ and **mean lifetime** $\bar{t} = 1/\lambda$
- The radioactivity of a sample is measured in **Curies** where $1 \text{ Ci} = 3.7{\times}10^{10} \text{ decay s}^{-1}$
- Radioactive nuclides can be used as clocks, in applications such as **carbon or geological dating**