HONOURS: GENERAL RELATIVITY WORKBOOK

Relativistic Physics

Class 1: Special Relativity

A) LORENTZ TRANSFORMATIONS

Einstein postulated that the speed of light is the same in all inertial reference frames, regardless of the motion of the source.

Consider two inertial reference frames S, recording events with space-time coordinates (ct, x, y, z), and S', with co-ordinates (ct', x', y', z'). Let's send a light signal out from the origin, when S and S' coincide. According to Einstein's postulate, events along the light signal must be related in S and S' by:

$$x^2 + y^2 + z^2 = (ct)^2$$
$$x'^2 + y'^2 + z'^2 = (ct')^2$$

Show that this requirement is satisfied in both frames if events transform from S to S' according to the Lorentz transformations:

$$ct' = \gamma(ct - \tfrac{v}{c}x)$$
$$x' = \gamma(x - \tfrac{v}{c}ct)$$
$$y' = y$$
$$z' = z$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

B) SPACE-TIME DIAGRAMS

Let's draw some space-time diagrams in frame S for events with space coordinate x and time coordinate t. On a graph of ct against x:

a) Draw the path of a light ray, and the path a particle travelling with speed $v < c$.

b) An event E occurs at $x = 0, t = 0$. Draw the locus of events in S which occur (i) 1 second of proper time after E, (ii) 1 second of proper time before E, (iii) 1 light-second of proper distance away from E.

Which of these events can be caused by E?

c) How do these loci of events look in the space-time diagram of ct' against x' in frame S'?

d) In the space-time diagram for frame S, draw the loci of events which occur at constant x', and at constant t', in the coordinate system of S'.

C) RELATIVISTIC MECHANICS

Conservation of Newtonian momentum $p = mv$ is inconsistent with special relativity. Here's a simple example to show why, and to illustrate the fix.

In frame S, consider two identical particles, A and B, of rest mass m_0 with equal and opposite velocities $\pm v$, colliding and sticking together to form a particle of mass $2m_0$. Now consider the collision as viewed from frame S', travelling with particle B.

a) In a Galilean transformation of velocities, what is the initial velocity of particle A in S'? Show that momentum $p = m_0 v$ is conserved in frame S'.

b) In a Lorentz transformation of velocities, what is the initial velocity of particle A in S'? [You will need to use the "addition of velocities" formula: $u' = (u + v)/(1 + \frac{uv}{c^2})$]. Show that momentum $p = m_0 v$ is not conserved in S', but we do conserve momentum if we modify the definition of mass to depend on velocity such that

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

D) “PARADOX” OF SPECIAL RELATIVITY

Analysis of events in special relativity can be illustrated by certain apparent “paradoxes”. A famous example is the “twin paradox”, in this activity we consider another example.

A barn has proper length L. A pole, also of proper length L, is carried towards the barn by a fast-moving runner. In the rest frame of the barn, S, the pole is observed as contracted to length L/γ, so should fit inside the barn. However, in the runner’s frame, S', the barn appears contracted to length L/γ, so the pole cannot fit inside.

Explain why this situation is not a paradox by drawing space-time diagrams in S and S', marking in 4 events:

E_1: the front end of the pole passes the front door of the barn
E_2: the front end of the pole passes the rear door of the barn
E_3: the rear end of the pole passes the front door of the barn
E_4: the rear end of the pole passes the rear door of the barn

Using your space-time diagrams, in what order do these events occur in S and S'?

Class 2: Index Notation

A) PRODUCING 4-VECTORS

A 4-vector is a group of four physical quantities whose values in different inertial frames are related by the Lorentz transformations. The prototypical 4-vector is the space-time coordinates of an event $x^\mu = (ct, x, y, z)$.

The sum or difference of two 4-vectors is also a 4-vector. Hence, taking the difference between two neighbouring events, we find the 4-vector $dx^\mu = (cdt, dx, dy, dz)$. New 4-vectors may also be produced by multiplying or dividing by an invariant.

a) Divide dx^μ by the invariant proper time interval $d\tau$ to obtain the components of the 4-velocity of a particle $v^\mu = dx^\mu/d\tau$ in terms of its velocity $\vec{v} = (v_x, v_y, v_z)$.

b) By applying the Lorentz transformations to the 4-velocity, find a relation between the x-components of velocity of a particle in frames S and S'.

c) Multiply v^μ by the invariant rest mass m_0 to obtain the components of the 4-momentum of a particle $p^\mu = m_0 v^\mu$ in terms of its energy E and momentum $\vec{p} = (p_x, p_y, p_z)$.

d) Now consider applying the results of parts b) and c) to a photon, which has zero rest mass. If $v_x = c$, what is v_x'? What is p^μ for a photon?

B) INDEX NOTATION PRACTICE

We introduce the "down 4-vector" with lowered index, where we change the sign of the first component, such that $x_\mu = (-ct, x, y, z)$. We can then write the space-time interval as

$$ds^2 = \sum_{\mu=0}^{3} dx_\mu dx^\mu$$

In index notation we don't write the summation, so this equation reads $ds^2 = dx_\mu dx^\mu$. Whenever we have a pair of raised/lowered indices, a sum over that index is implied.

The process of converting from an "up" to a "down" 4-vector can be written as $x_\mu = \eta_{\mu\nu} x^\nu$, where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Likewise, $x^\mu = \eta^{\mu\nu} x_\nu$, where $\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) The Lorentz transformations may be written $x'^\mu = L^\mu{}_\nu x^\nu$. What is the matrix $L^\mu{}_\nu$?

b) Write an expression in index notation for the inverse Lorentz transformations of an up 4-vector. What matrix carries out the transformation?

c) What is the matrix $L^{\mu\lambda} = \eta^{\lambda\nu} L^\mu{}_\nu$?

d) We have seen that $dx_\mu dx^\mu$ is an invariant. What are the values of the invariant quantities $v_\mu v^\mu$, $p_\nu p^\nu$, $v_\alpha p^\alpha$, $\eta_{\kappa\lambda}\eta^{\kappa\lambda}$ and $L^{\alpha\beta} L_{\alpha\beta}$?

C) 4-CURRENT AND CONSERVATION LAWS

The space-time volume element $dV\ dt$ is a Lorentz invariant (since $dx' = \gamma\ dx$ and $dt' = dt/\gamma$, then $dx'\ dt' = dx\ dt$). If a small region of space-time contains electric charge dQ, we may hence construct a 4-vector,

$$J^\mu = \frac{dQ\ dx^\mu}{dV\ dt}$$

a) Let dx^μ represent the space-time displacement of all the charges in the region, in some small interval. Use the components of x^μ, and the definition of current, to show that the components of this 4-vector are $J^\mu = (\rho c, J_x, J_y, J_z)$ in terms of charge density ρ and spatial current density $\vec{J} = (J_x, J_y, J_z)$.

b) Charge conservation in electromagnetism is expressed by $\vec{\nabla}.\vec{J} = -\partial\rho/\partial t$. Show that this relation may be written in index notation as $\partial_\mu J^\mu = 0$.

D) ENERGY-MOMENTUM TENSOR

Now suppose a small region of space-time contains momentum dp^{μ}. We define the energy-momentum tensor as,

$$T^{\mu\nu} = \frac{dp^{\mu}\, dx^{\nu}}{dV\, dt}$$

As in Activity C, suppose dx^{μ} represents the space-time displacement of all the matter-energy in the region, in some small interval.

a) Use the components of p^{μ} and x^{μ} to show that T^{00} represents the energy density in this region.

b) Show that T^{0i} is the flux of energy in the x^{i}-direction $(i > 0)$.

c) Show that T^{ij} is the flux of i-momentum in the x^{j}-direction $(i, j > 0)$.

Class 3: Electromagnetism

A) MAXWELL'S EQUATIONS RE-VISITED

Electromagnetism may be described in terms of the Maxwell field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

In this equation, $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ – to obtain ∂^μ you would raise the index – and $A^\mu = \left(V/c, A_x, A_y, A_z\right)$ is the electromagnetic potential 4-vector, where V is the electrostatic potential and $\vec{A}$ is the magnetic vector potential. Substituting in the relations for the electric field $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$ and magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$, we find:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

a) In tensor notation, two of Maxwell's Equations can be written compactly as

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$$

where μ_0 is the permeability of free space, and $J^\mu = (\rho c, J_x, J_y, J_z)$ is the current 4-vector. By considering cases $\nu = 0$ and $\nu = 1$, show that you recover two of Maxwell's equations.

b) Show that $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$.

c) By considering cases $(\lambda, \mu, \nu) = (0,1,2)$ and $(1,2,3)$ in the relation in part b), show that you recover the other two of Maxwell's Equations.

B) MAGNETIC FIELD OF A CURRENT

By applying the Lorentz transformation to the Maxwell field tensor, we can deduce how electromagnetic fields transform between two frames S and S':

$$F'^{\mu\nu} = L^{\mu}{}_{\kappa} L^{\nu}{}_{\lambda} F^{\kappa\lambda}$$

where $L^{\mu}{}_{\kappa} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

a) Use the Lorentz transformation to show that:

$$B'_x = B_x$$
$$B'_y = \gamma\left(B_y + vE_z/c^2\right)$$
$$B'_z = \gamma\left(B_z - vE_y/c^2\right)$$

b) Consider a static line of charge in frame S, such that $\vec{B} = \vec{0}$. Gauss's Law shows that $E_y = \lambda/2\pi\varepsilon_0 y$ (at $z = 0$) in S, where λ is the charge per unit length.

In frame S', the line of charge becomes a current. Use the Lorentz transformation to recover the expected magnetic field strength at distance d from a current I, which at $z = 0$ is $B_z{}' = \mu_0 I/2\pi y$.

C) ELECTROMAGNETIC ENERGY DENSITY AND FLOW

The energy-momentum tensor $T^{\mu\nu}$ for electromagnetism is:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\lambda}F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}\right)$$

Recall that $F^{\mu}{}_{\lambda} = \eta_{\lambda\nu}F^{\mu\nu}$ and $F_{\kappa\lambda} = \eta_{\kappa\mu}\eta_{\lambda\nu}F^{\mu\nu}$, where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Let's first consider the energy density component, T^{00}.

a) Show that $F_{\kappa\lambda}F^{\kappa\lambda} = 2(B^2 - E^2/c^2)$ and $F^{0}{}_{\lambda}F^{0\lambda} = E^2/c^2$.

b) Hence show that the energy density in electromagnetic fields is $T^{00} = \frac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0$.

Now consider the flow of energy in each direction, T^{0i}.

c) Show that $T^{0x} = (E_y B_z - E_z B_y)/\mu_0 c$.

This is the x-component of the Poynting vector $\frac{1}{\mu_0}\frac{\vec{E}\times\vec{B}}{c}$.

Class 4: Accelerated Motion

A) WORLD-LINE OF ACCELERATING OBJECT

Consider an object moving with constant proper acceleration α. This means that

$$\alpha = \frac{dv\prime}{d\tau} = \text{constant}$$

where $d\tau$ is the proper time elapsed in a small interval, and dv' is the momentary increase in speed from rest in S'. Assume the object is at rest in S at $\tau = 0$.

a) Use the relativistic addition of velocities formula to show that the increase in velocity in S is $dv \approx dv'\left(1 - \frac{v^2}{c^2}\right)$.

b) Hence by substituting in $dv' = \alpha\, d\tau$, show that $v = c \tanh\left(\frac{\alpha\tau}{c}\right)$ at proper time τ.

c) Use $\gamma = 1/\sqrt{1 - v^2/c^2} = \cosh\left(\frac{\alpha\tau}{c}\right)$, and the time interval in S, $dt = \gamma\, d\tau$, to show that the time coordinate t in S is related to the proper time τ by $t = \frac{c}{\alpha}\sinh\left(\frac{\alpha\tau}{c}\right)$.

d) Starting from the relation for the space-time interval, for $d\tau$ in terms of dt and dx, show that the x-coordinate of the object in S is given by $x = \frac{c^2}{\alpha}\cosh\left(\frac{\alpha\tau}{c}\right)$.

e) Draw the world line of the object in S on a space-time diagram of ct against x.

Gravity and Curvature

Class 5: Equivalence Principle

A) GRAVITATIONAL BENDING OF LIGHT

A consequence of the Equivalence Principle is that light will be bent in a gravitational field. How much bending should we see in a laboratory at rest on the Earth's surface? According to the Equivalence Principle, in such a laboratory one would observe the same effects as in a laboratory accelerating in deep space with a uniform acceleration of $g = 9.8\ m\ s^{-2}$.

Imagine that a laser at one end of the laboratory emits a beam of light that originally travels parallel to the laboratory floor. The light shines on the opposite wall of the laboratory, at a horizontal distance of $d = 3.0\ m$.

a) What is the magnitude of the vertical deflection of the light beam?

b) What is the magnitude of this deflection if the laboratory sits on the surface of a neutron star, which has a mass $M = 3.0 \times 10^{30}\ kg$ and radius $R = 12\ km$? (For the purposes of this question, neglect strong-field effects and calculate g using Newtonian methods!)

B) THE GLOBAL POSITIONING SYSTEM

The Global Positioning System (GPS) is a network of satellites that allows anyone, with the aid of a small device (receiver), to determine exactly where they are on the Earth's surface. Each satellite contains a very precise clock and microwave transmitter.

a) Suppose the clocks on the GPS satellites contain a very small error, such that they drift by "only" 1 part in 10 billion. What distance error would accumulate every day?

The proper time interval $d\tau$ between 2 events, in terms of the co-ordinate time interval dt, is $d\tau = dt\sqrt{1 + 2\phi/c^2}$, where ϕ is the gravitational potential.

b) Assuming the weak field expression for the gravitational potential near the Earth, $\phi = -GM/r$, and considering for the moment that the clocks are at rest in the gravitational field, what fractional timing error is caused by the difference in ϕ between the Earth's surface and the satellites? (Estimate or look up the data you need). Do the satellite clocks run fast or slow compared to Earth clocks?

c) The GPS satellites are in motion, orbiting the Earth. For the purposes of this part of the question we will assume that the satellites and Earth observers are in the same inertial reference frame. Estimate the velocity of the satellites in their orbit, and hence use time dilation in Special Relativity to determine the fractional timing error caused by the motion of the satellites in orbit. Do the satellite clocks run fast or slow compared to Earth clocks?

Class 6: Curved Space and Metrics

A) GEOMETRY ON A CURVED SURFACE

The normal geometric relations in flat space do not apply in a curved space. Consider a 2D spherical surface with co-ordinates (θ, ϕ). To make it easy to visualize, we'll consider the surface of a 3D sphere of radius R.

a) Show that the distance metric on the surface of the sphere is

$$ds^2 = R^2 \, d\theta^2 + (R \sin\theta)^2 \, d\phi^2$$

b) Starting from the North Pole, move a small constant distance ε in all directions, forming a "circle" in the curved space. Show that the circumference of this circle is not the flat-space relation $2\pi\varepsilon$, but rather,

$$C \approx 2\pi\varepsilon \left(1 - \frac{\varepsilon^2}{6R^2}\right)$$

c) The area element of a 2D co-ordinate space with metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is $dA = \sqrt{|g|} \, dx^0 dx^1$. Using the metric of part a), show that the area element of a spherical surface is $dA = R^2 \sin\theta \, d\theta d\phi$.

d) Show that the area of the circle in part b) is not the flat-space relation $\pi\varepsilon^2$, but rather,

$$A \approx \pi\varepsilon^2 \left(1 - \frac{\varepsilon^2}{12R^2}\right)$$

B) METRICS IN 2D

The metric determines the geometry of space. But the geometry does not uniquely determine the metric, because we may always transform co-ordinates.

a) What are some geometrical methods we could use to determine whether a given co-ordinate space is flat or curved?

b) Motivated by the result of Activity A, part d), we can define the curvature of a 2D surface at a point by the relation

$$k = \lim_{\varepsilon \to 0} \left[\frac{12}{\varepsilon^2} \left(1 - \frac{A}{\pi \varepsilon^2} \right) \right]$$

where A is the area enclosed by moving a small constant distance ε. What is the curvature of the 2D spherical surface from Activity A?

c) Consider two distance metrics for co-ordinates (r, θ). The first is a polar co-ordinate system, with $ds^2 = dr^2 + r^2 d\theta^2$. The second is a modified co-ordinate system with metric

$$ds^2 = \frac{dr^2 + r^2 d\theta^2}{1 + r^2}$$

Use the formula in part b) to find the curvature at $r = 0$ of these two spaces. Do they represent flat or curved space?

Class 7: Geodesics

A) GEODESICS ON A SPHERE

As in Class 6, we'll consider a 2D spherical surface with co-ordinates (θ, ϕ), by embedding a sphere of radius R in a 3D space.

a) Write down the metric elements $g_{\theta\theta}$, $g_{\phi\phi}$, $g_{\theta\phi}$ and $g_{\phi\theta}$ on the surface of the sphere.

b) What are the values of $g^{\theta\theta}$ and $g^{\phi\phi}$?

c) Use the relation for the Christoffel symbols, $\Gamma^{\mu}_{\kappa\lambda} = \frac{1}{2} g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$, to show that the non-zero symbols are $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$.

d) Hence show that the geodesic equations $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$ on the surface are:

$$\frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0 \qquad\qquad \frac{d^2\phi}{d\tau^2} + 2\cot\theta\left(\frac{d\theta}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) = 0$$

e) Consider the geodesic between two points A and B on the sphere. Without loss of generality, we can rotate the coordinate system such that the two points are on the equator, $\theta = \pi/2$. In this case, find the geodesic and explain why it is a "great circle".

B) MOTION IN A WEAK FIELD

The space-time metric of a weak, static gravitational field is

$$g_{\mu\nu}(x^i) = \eta_{\mu\nu} + h_{\mu\nu}(x^i)$$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the metric for flat space-time, and $|h_{\mu\nu}| \ll 1$ is a small perturbation which depends only on spatial co-ordinates $x^i = (x, y, z)$, not time.

a) Particles move along geodesics which satisfy $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$. If a particle is slowly moving, then $\frac{dx^i}{d\tau} \ll \frac{dx^t}{d\tau}$. Explain why this implies that, in terms of co-ordinate time t,

$$\frac{d^2x^\mu}{dt^2} \approx -c^2\,\Gamma^\mu_{tt}$$

b) Use the relation for the Christoffel symbols, $\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$, to show that, for $i = (x, y, z)$,

$$\Gamma^i_{tt} \approx -\frac{1}{2}\frac{\partial h_{tt}}{\partial x^i}$$

c) Newton's Laws relate the acceleration of a particle to the gravitational potential $\phi(\vec{x})$ via $\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi$. Use the results for parts a) and b) to demonstrate that the weak-field metric is

$$g_{tt} \approx -1 - \frac{2\phi}{c^2}$$

d) Hence for a clock at rest in a weak gravitational field, show that a co-ordinate time interval dt is related to the proper time interval $d\tau$ by

$$dt = \frac{d\tau}{\sqrt{1 + 2\phi/c^2}}$$

Class 8: Space-time Geometry

A) RIEMANN TENSOR ON A SPHERE

In this Activity, we will compute as an example the Riemann curvature tensor on a 2D spherical surface, with metric:

$$ds^2 = R^2\, d\theta^2 + (R\sin\theta)^2\, d\phi^2$$

The Riemann tensor may be determined from the Christoffel symbols using the relation,

$$R^{\kappa}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\lambda\nu} - \partial_\nu \Gamma^{\kappa}_{\lambda\mu} + \Gamma^{\kappa}_{\mu\alpha}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\kappa}_{\nu\alpha}\Gamma^{\alpha}_{\lambda\mu}$$

In the previous class, we saw that the only non-zero Christoffel symbols for this metric are $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$.

a) For a 2D space, the Riemann tensor has only 1 independent component. Show that this component may be written

$$R^{\theta}{}_{\phi\theta\phi} = (\sin\theta)^2$$

b) Use the relation for the Riemann tensor in terms of the Christoffel symbols to show that

$$R^{\phi}{}_{\theta\phi\theta} = 1$$

c) Since there is only 1 independent component, we must be able to deduce $R^{\phi}{}_{\theta\phi\theta}$ from $R^{\theta}{}_{\phi\theta\phi}$! We can show from the definition of the Riemann tensor that two symmetries are:

$$R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\mu\nu} \qquad R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu}$$

Use these symmetries to deduce the result of part b) from part a).

Black Holes and the Universe

Class 9: Black Holes

A) THE SCHWARZSCHILD RADIUS

a) The Schwarzschild radius of an object of mass M is $R_S = 2GM/c^2$. A black hole is an object which has a radius $r < R_S$. Determine the minimum density of an object which satisfies this requirement if (1) $M = 1\ M_\odot$, (2) $M = 10^{10}\ M_\odot$.

b) In Class 4 we related the change in the clock rate C with proper distance L to the proper acceleration α, which is equivalent to the gravitational field.

$$\frac{dC}{C} = \frac{\alpha\, dL}{c^2}$$

In the Schwarzschild metric the clock rate $C \propto \sqrt{1 - R_S/r}$, and proper distance interval dL is related to co-ordinate distance interval dr as $dL = dr/\sqrt{1 - R_S/r}$. Show that:

$$\alpha = -\frac{GM}{r^2\sqrt{1 - R_S/r}}$$

What are the values of α at $r \gg R_S$ and $r = R_S$?

B) RADIAL PLUNGE INTO A BLACK HOLE

The Schwarzschild space-time metric around a black hole is

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{R_S}{r}} + r^2 [d\theta^2 + (\sin\theta)^2 d\phi^2]$$

in terms of the Schwarzschild radius R_S. Freely-falling observers have world-lines $x^\mu(\tau)$ following geodesics $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda} \frac{dx^\kappa}{d\tau} \frac{dx^\lambda}{d\tau} = 0$, where $\Gamma^\mu_{\kappa\lambda} = \frac{1}{2} g^{\mu\nu} (\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$.

a) Writing $A = 1 - R_S/r$, show that $\Gamma^t_{rt} = \frac{1}{2A}\frac{dA}{dr}$. Hence demonstrate that the $\mu = t$ geodesic equation may be written in the form

$$\frac{d}{d\tau}\left(A \frac{dt}{d\tau}\right) = 0$$

and hence that $dt/d\tau = K/A$, where K is a constant.

b) Writing $ds^2 = -c^2 d\tau^2$, use the original equation for the metric to demonstrate that, for an object radially plunging into a black hole (such that $d\theta = d\phi = 0$),

$$-A\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{Ac^2}\left(\frac{dr}{d\tau}\right)^2 + 1 = 0$$

c) Consider an object which is at rest ($dr/d\tau = 0$) at $r = \infty$. What is the value of K? Show that the proper time required to travel from $r = R_0$ to $r = 0$ is

$$\Delta\tau = \frac{2{R_0}^{3/2}}{3c{R_s}^{1/2}}$$

d) What is the co-ordinate time interval Δt required to reach $r = R_S$?

C) ORBITS AROUND A BLACK HOLE

a) Light rays move through space-time such that $ds = 0$. Use the Schwarzschild metric to show that for a radially-moving light ray near a black hole,

$$\frac{1}{c}\frac{dr}{dt} = 1 - \frac{R_S}{r}$$

Why does this equation imply that a light ray emitted from $r < R_S$ cannot escape the black hole? What happens to a light ray emitted at $r = R_S$?

Now consider a light ray in a circular orbit around a black hole, such that $r =$ constant. We can choose the orbit in the ϕ direction, such that $\theta = 90°$.

b) Use the condition $ds = 0$ for this orbit to show that the angular velocity of the light ray is

$$\frac{1}{c}\frac{d\phi}{dt} = \frac{\sqrt{1 - R_S/r}}{r}$$

c) Use the $\mu = r$ component of the geodesic equation $\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{dp}\frac{dx^\lambda}{dp} = 0$ to show that

$$c^2\,\Gamma^r_{tt}\left(\frac{dt}{dp}\right)^2 + 2c\,\Gamma^r_{t\phi}\left(\frac{dt}{dp}\right)\left(\frac{d\phi}{dp}\right) + \Gamma^r_{\phi\phi}\left(\frac{d\phi}{dp}\right)^2 = 0$$

d) Use the results $\Gamma^r_{tt} = \frac{1}{2}A\frac{dA}{dr}$, $\Gamma^r_{t\phi} = 0$ and $\Gamma^r_{\phi\phi} = -Ar(\sin\theta)^2$, in terms of $A = 1 - R_s/r$, to show that we obtain a second relation for the angular velocity,

$$\frac{1}{c}\frac{d\phi}{dt} = \sqrt{\frac{R_S}{2r^3}}$$

e) By equating the results of parts b) and d), find the radius of orbit of light rays in a circular orbit around a black hole.

Class 10: Einstein equation

A) THE NEWTONIAN LIMIT

In Class 7, Activity B, we showed that the first entry of the space-time metric for a weak gravitational field was $g_{tt} \approx -1 - 2\phi/c^2$, in terms of Newtonian gravitational potential $\phi(x_i)$. We also calculated the Christoffel symbol $\Gamma^i_{tt} \approx \frac{1}{c^2}\frac{\partial \phi}{dx^i}$.

The Ricci tensor $R_{\mu\nu}$ is related to the Christoffel symbols by

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\kappa_{\kappa\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\kappa}$$

a) Show that $R_{tt} \approx \nabla^2\phi/c^2$. (Hint: since this is a weak field, we can neglect the last 2 terms because they are products of small quantities).

b) The Einstein equation relates space-time curvature to matter-energy by $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$, where the Ricci scalar R may be calculated as $R = g^{\mu\nu}R_{\mu\nu}$. By applying $g^{\mu\nu}$ to both sides of the equation, show that the Einstein equation may be re-written in the form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

where $T = g^{\mu\nu}T_{\mu\nu}$.

c) Explain why the matter-energy tensor $T_{\mu\nu}$ for slowly-moving matter with mass density ρ is $T_{tt} \approx \rho c^2, T_{rest} \approx 0$. Hence show that, for a weak field, $T \approx -\rho c^2$.

d) Use the above results to demonstrate that, in the weak-field limit, the Einstein equation is consistent with the Newtonian relation for the gravitational potential, $\nabla^2\phi = 4\pi G\rho$.

Class 11: Cosmology

A) LIGHT RAYS IN EXPANDING SPACE

The metric of a homogeneous Universe of curvature k, expanding with scale factor $a(t)$, in terms of space-time co-ordinates (t, r, θ, ϕ) is:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + (\sin\theta)^2 d\phi^2) \right]$$

a) Use the relation for the Christoffel symbols $\Gamma^{\mu}_{\kappa\lambda} = \frac{1}{2} g^{\mu\nu} (\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$ to show that for this space-time metric,

$$\Gamma^{t}_{rr} = \frac{\dot{a}}{a} \frac{g_{rr}}{c}$$

(the other symbols $\Gamma_{rest} = 0$).

b) Light rays with 4-vector $k^\mu = dx^\mu / dp$ satisfy the geodesic equation $\frac{dk^\mu}{dp} + \Gamma^{\mu}_{\kappa\lambda} k^\kappa k^\lambda = 0$ and the relation for zero space-time interval, $g_{\mu\nu} k^\mu k^\nu = 0$. Use these relations together with the result from part a) to show that, for a radially propagating light ray,

$$\frac{dk^t}{dp} + \frac{\dot{a}}{a} (k^t)^2 = 0$$

c) The frequency ω of the light ray is given by $k^t = \omega / c$. Show that the result of part b) implies that the frequency of a light ray in an expanding Universe changes such that

$$\omega \propto \frac{1}{a}$$

B) THE FRIEDMANN EQUATION

The non-zero elements of the Ricci tensor $R_{\mu\nu}$ of the space-time metric in Activity A are:

$$R_{tt} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

$$R_{ii} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2kc^2}{a^2}\right]\frac{g_{ii}}{c^2}$$

a) Show that the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right]$$

b) Hence show that the first component of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is,

$$G_{tt} = \frac{3}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right]$$

c) Use the $\mu = t, \nu = t$ component of the Einstein equations, $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$, and the energy-momentum tensor for slowly-moving matter, $T^{tt} = \rho(t)c^2$ and $T^{rest} = 0$, to show that the scale factor of the expanding Universe satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{kc^2}{a^2}$$

C) LIGHT TRAVEL IN AN EXPANDING UNIVERSE

Let us combine the results of Activities A and B to determine, if a ray of light reaches our telescopes with redshift z, how long has it been travelling through the expanding Universe?

We'll suppose that the Universe today ($t = t_0$) has zero curvature ($k = 0$) and a special matter density called the critical density,

$$\rho(t_0) = \frac{3{H_0}^2}{8\pi G}$$

where H_0 is the value of $\dot{a}/a$ in today's Universe, known as the Hubble constant. The matter density at other scale factors is then,

$$\rho(t) = \frac{\rho(t_0)}{a^3}$$

a) Use the Friedmann equation to show that the evolution of the scale factor is governed by

$$\frac{da}{dt} = \frac{H_0}{\sqrt{a}}$$

b) Hence show that a ray of light with redshift z has been travelling through the Universe for co-ordinate time

$$t = \frac{2}{3H_0}\left[1 - \frac{1}{(1+z)^{3/2}}\right]$$

c) What is the radial co-ordinate of the object that emitted this light ray? Use the fact that light rays travel with $ds = 0$ to show that, in this Universe with zero curvature,

$$\frac{dr}{dt} = \frac{c}{a}$$

d) Hence show that

$$r = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$