# Cosmology Week 3 Class Activities

We'll study these activities in our Week 3 tutorial class!

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## A dark-energy dominated Universe

Suppose that dark energy is the only component of the Universe, such that the density parameters today are $\Omega_m = 0$ and $\Omega_\Lambda = 1$. (This is thought to be a good approximation for our own Universe, far into the future!)

a) Is the spatial geometry of this model universe saddle-like, flat or spherical?

b) Use the Friedmann equation to show that this model universe expands exponentially, such that

$$a(t) = e^{H_0(t-t_0)}$$

Where $H_0$ is Hubble's constant and $t_0$ is the time co-ordinate today.

c) How does the Hubble parameter vary with time in this model universe?

d) What is the age of this model universe, in terms of $H_0$?

Assume that $t_0 = 13.7$ Gyr, and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$. A ray of light is emitted in this model Universe at time $t = t_0/2$, and travels on a radial path to reach $r = 0$ today.

e) At what co-moving co-ordinate $r$ was the light emitted, in units of Gpc?

f) What is the redshift of the light when it reaches $r = 0$?

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## The coincidence problem

In today's Universe, the density parameters are $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. The Universe will transition from its expansion being dominated by matter at early times to dark energy at late times.

a) At matter-dark energy equality, $z_{eq}$, the energy density of the matter and dark energy components is equal, $\rho_m(z_{eq}) = \rho_\Lambda(z_{eq})$. What is the value of $z_{eq}$?

b) Write down the Friedmann equation for the scale factor $a(t)$ in a flat Universe, in terms of $\Omega_m$, $\Omega_\Lambda$ and $a$.

c) By differentiating this expression, find an expression for the "acceleration" of the expansion, $\ddot{a} = \frac{d^2a}{dt^2}$. At what redshift does $\ddot{a} = 0$?

The fact that the transition between matter domination and dark energy domination is happening relatively close to our existence in the Universe is sometimes called the coincidence problem.

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# A phantom Universe!

The equation of state of a substance relates the pressure $P$ it exerts to its energy density $\rho c^2$ by:

$$P = w\,\rho c^2$$

Imagine that we discover a weird, new component within the Universe called phantom energy, which has equation of state $w = -2$ ! (since we have already found a component with $w = -1$, maybe it isn't so weird!)

We showed in the class slides that conservation of energy for an expanding substance implies:

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0$$

(this is another of those cases where our "simple" calculation miraculously recovers the exact result from General Relativity!)

a) Show that this phantom energy component with $w = -2$ evolves such that $\rho(t) \propto a^3$.

b) The phantom energy component will grow to dominate the Universe. Using the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\,\rho(t)}{3}$$

how does the scale factor evolve with time into the future, if $a = 1$ at $t = t_0$ (today)?

c) Show that this Universe has a finite lifetime. How would this Universe appear to observers within it, as this lifetime approaches?

# Solving the general Friedmann equation

*This is a computer-based activity which I recommend we carry out using a Python Jupyter notebook (as widely used in astronomy research) – however, it can be solved using any programming language.*

The Friedmann equation including density parameters for matter ($\Omega_m$), curvature ($\Omega_K$) and dark energy ($\Omega_\Lambda$) is:

$$\frac{1}{{H_0}^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{H(a)^2}{{H_0}^2} = \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda$$

where $\Omega_m + \Omega_K + \Omega_\Lambda = 1$. Re-arranging the definition $H(a) = \frac{da/dt}{a}$, the cosmic time may be found by solving the integral:

$$t - t_0 = \int_1^a \frac{1}{a\,H(a)}\,da$$

Write a computer program to recreate the plot we have been considering in class: