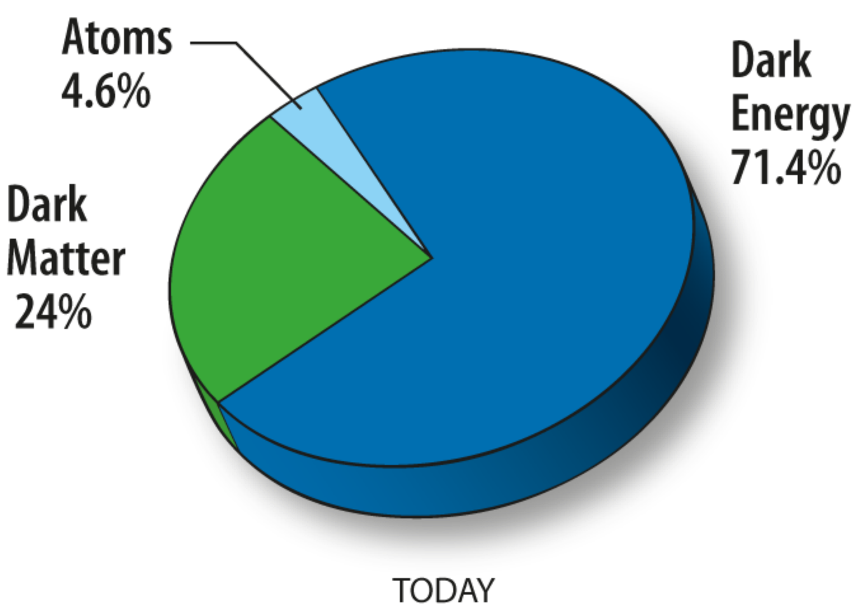
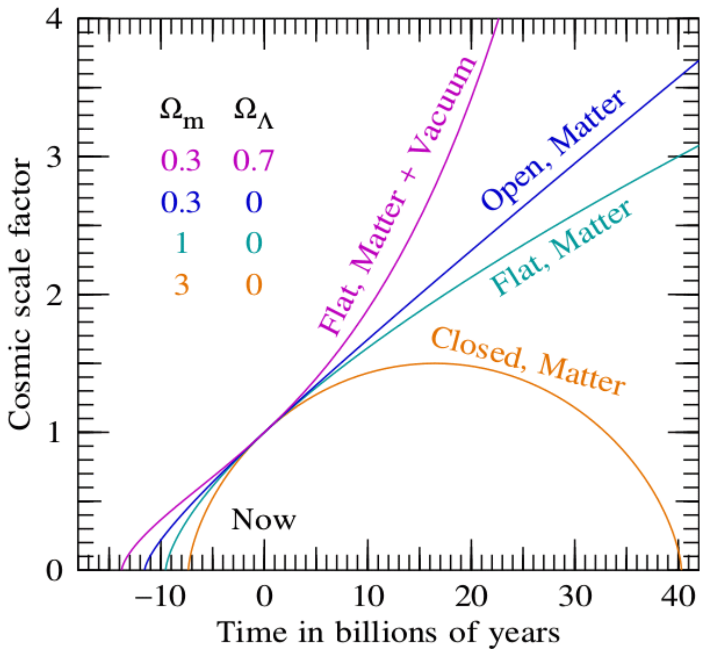
Honours Cosmology Week 3:
Contents of the Universe

In this class we will formulate the relations between the expansion of the Universe and the matter, radiation and dark energy it contains





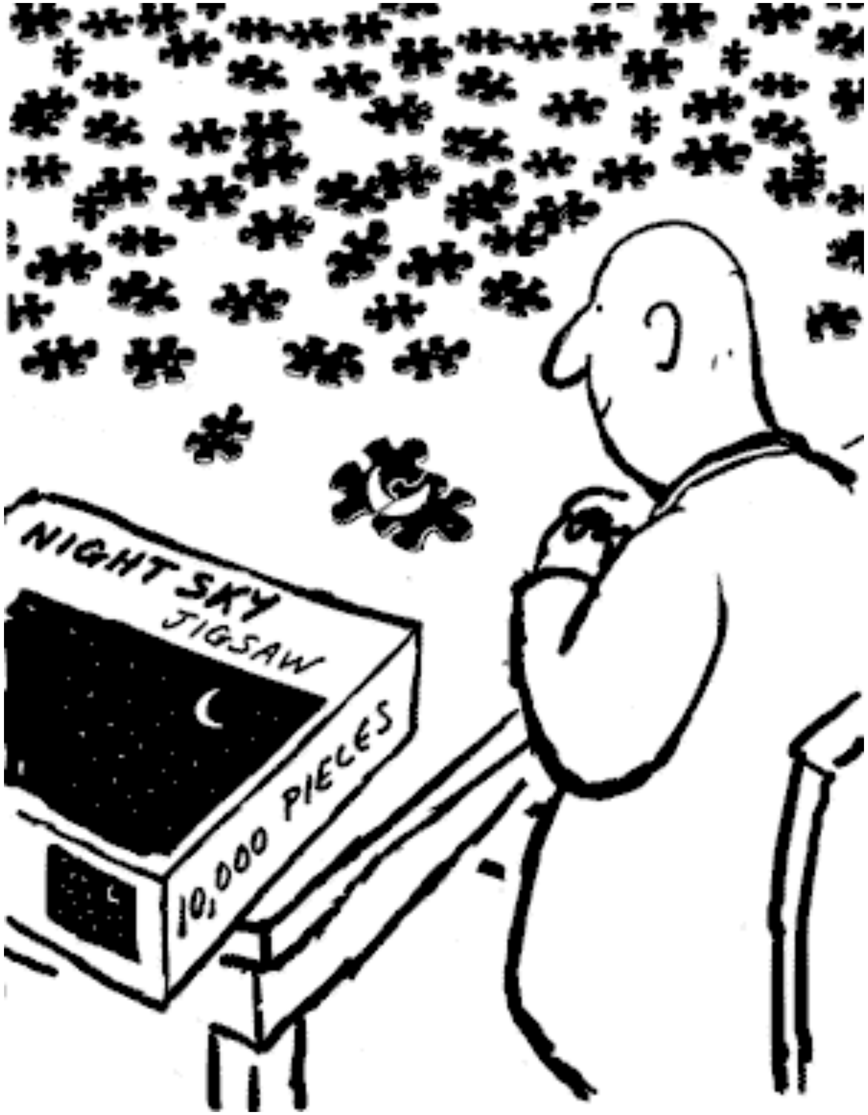
Contents of the Universe

At the end of this week you should be able to …

- … describe the components of energy in the Universe, including the properties of **dark matter** and **dark energy**
- … link these components to the cosmic expansion rate using the **Friedmann equation** and **energy conservation equation**
- … analyse these constituents using **density parameters**, Ω
- … solve for the relations between distance, redshift and time in **single-component models**
- … express components using their **equation of state**, w

Contents of the Universe

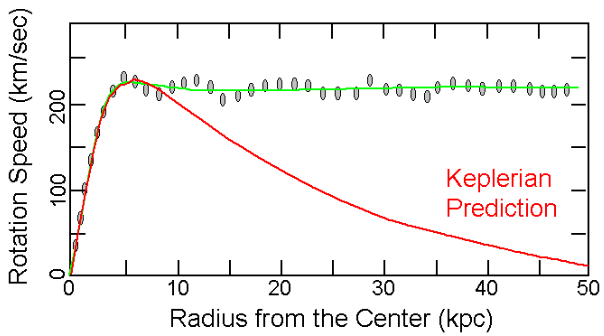
- In GR, *matter tells spacetime how to curve*
- Therefore, in order to understand expanding space, we need to understand its contents
- Let's start with *dark matter* – the majority of matter in the Universe that doesn't emit light



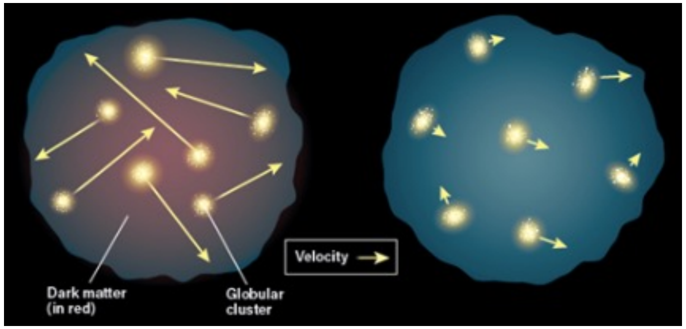
Dark matter

- You will have already heard about the *abundant observational evidence for dark matter*, including:

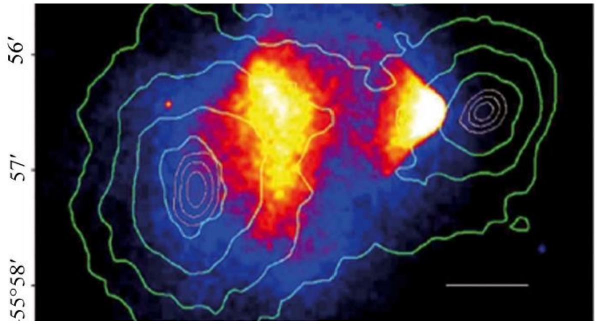
Galaxy rotation curves



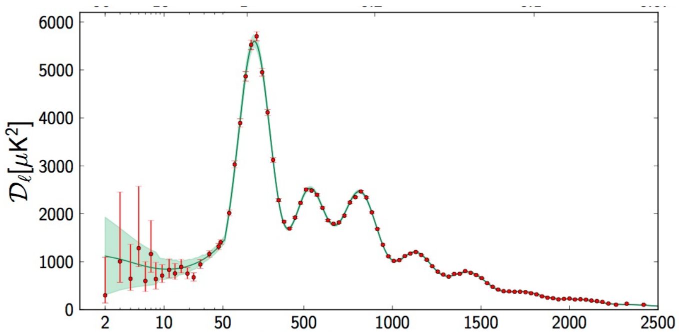
Cluster velocity dispersions



Gravitational lensing



CMB physics

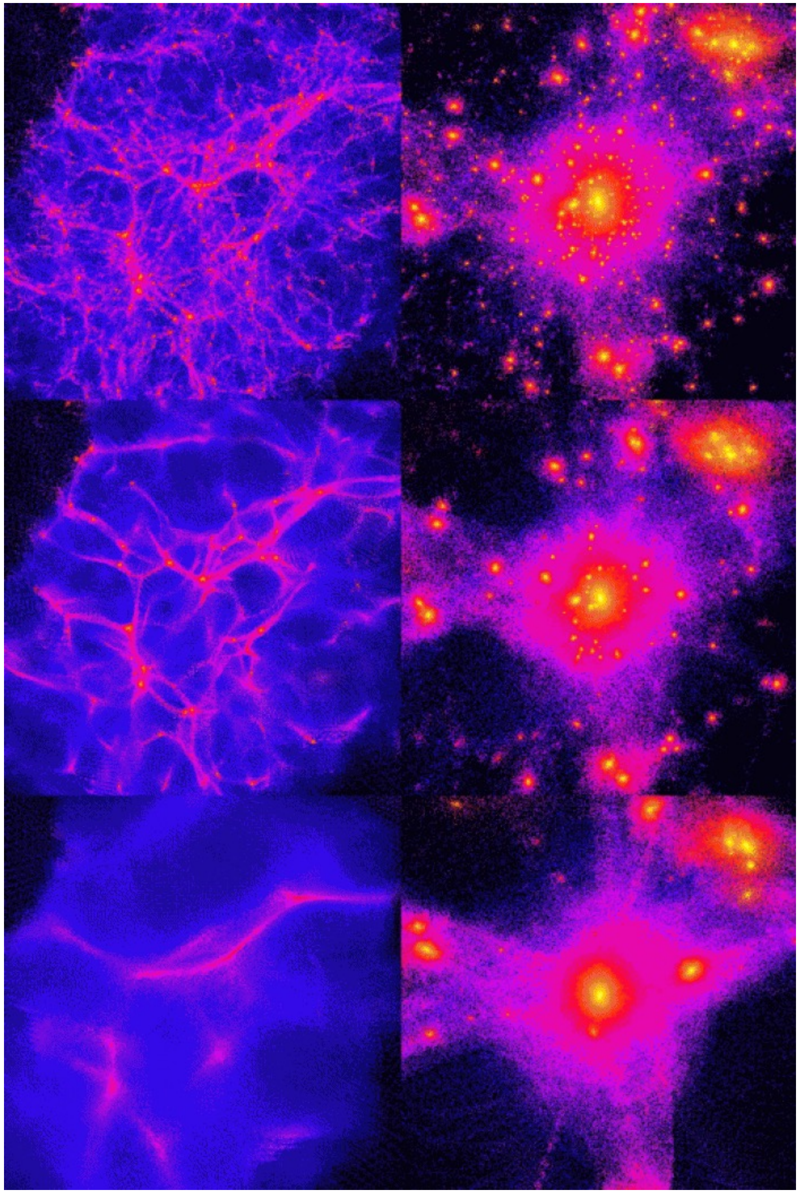


Dark matter

Observations and models tell us that dark matter is:

- *Weakly-interacting*
- *Non-baryonic*
- *Cold (i.e. slowly-moving)*

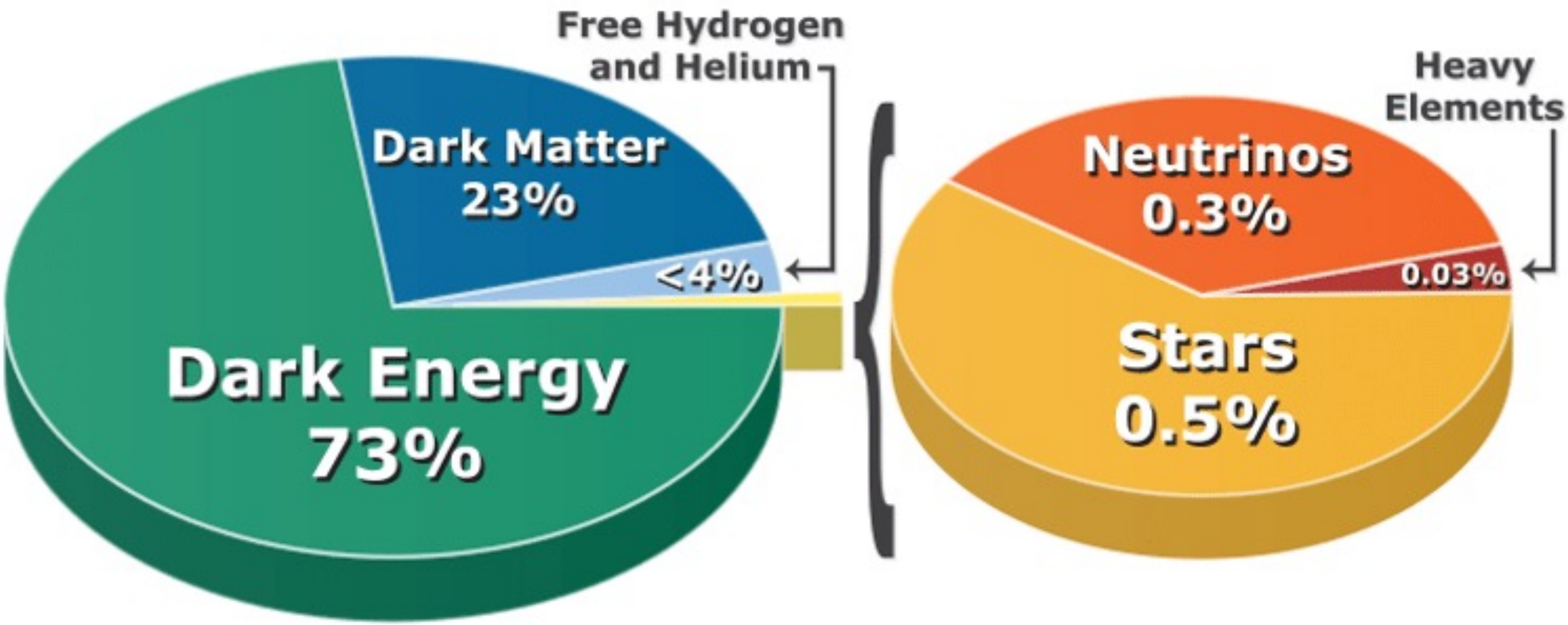
However, there is no clear candidate within the standard model of particle physics!



Dark energy

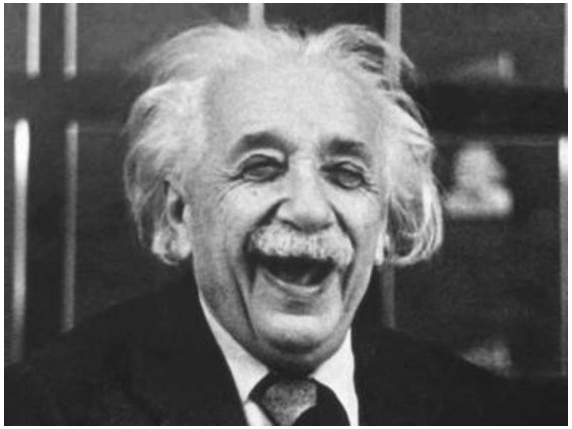
- Baryonic and dark matter alone **cannot explain** the expansion of the Universe, because we have also observed that it is accelerating (more on this later)

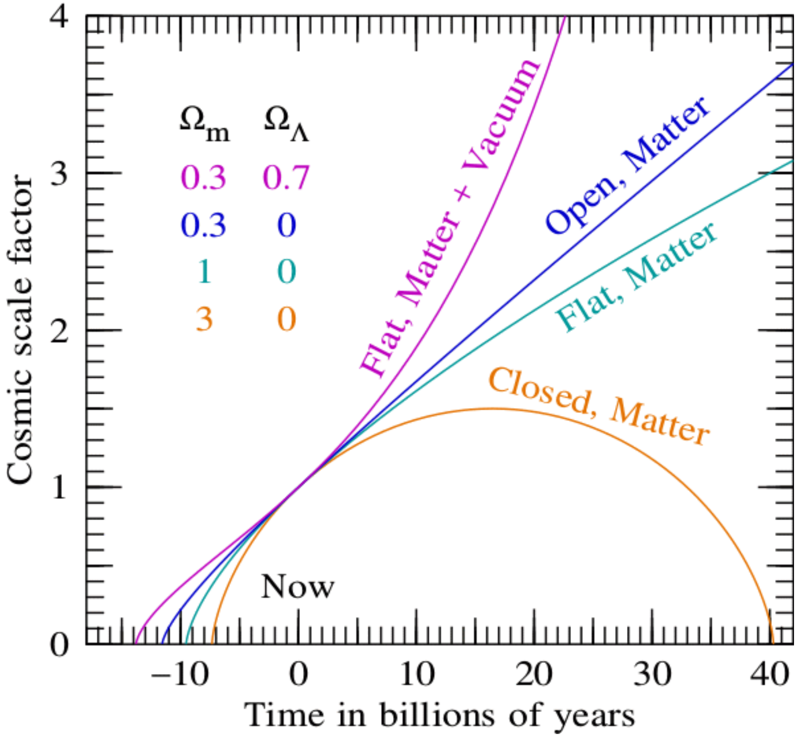
- We turn to a new component – *dark energy*



Dark energy

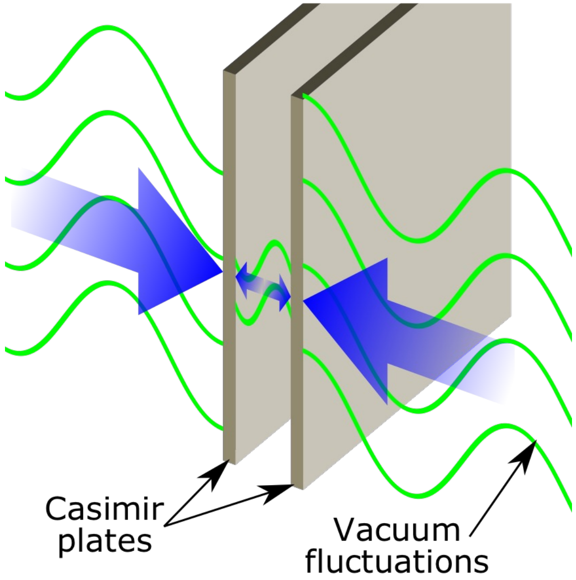
- Dark energy is an *"anti-gravity" effect* originally introduced by Einstein
- At this time, which pre-dated any observations, Einstein favoured a "static Universe" which required a component of this form to cancel out gravitational attraction
- The dark energy field is commonly known as the *cosmological constant, Λ*

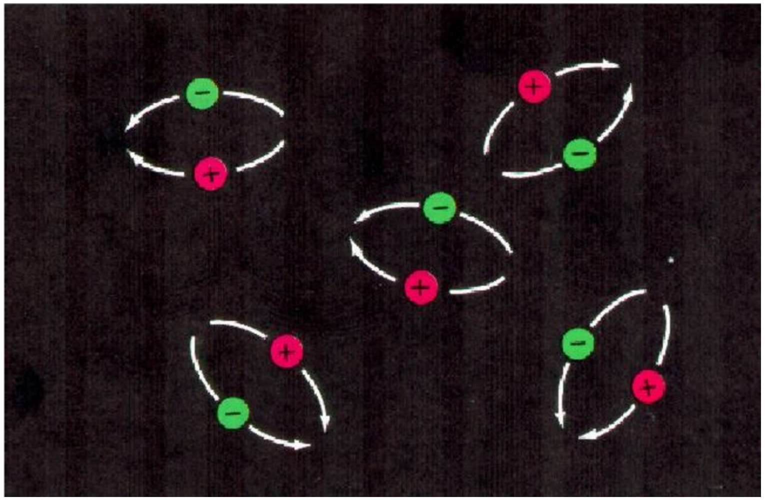




Dark energy

- An effect like Λ is produced by the zero-point energy of the quantum vacuum (see: Casimir effect)
- It's familiar in physics that the overall zero-point of energy can be changed without affecting the evolution of a system
- But in GR all energy gravitates, so a zero-point energy still contributes to cosmic expansion

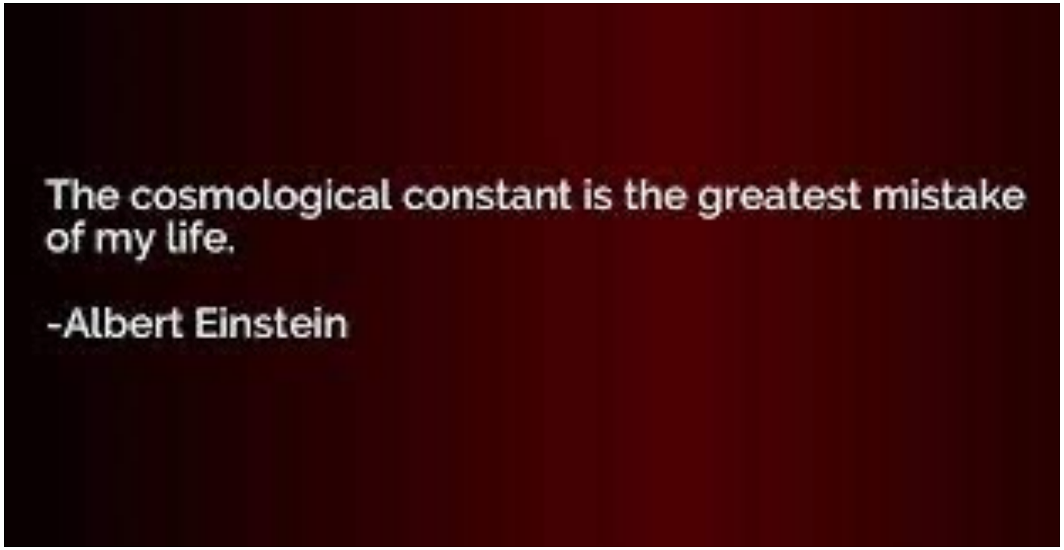




Dark energy

- However, if Λ is really generated by quantum fluctuations, its predicted value differs **very greatly** from that measured by cosmological observations

- Another strange aspect: as Λ represents a constant energy density, energy is being created as the Universe expands!



The Friedmann equation

- Now let's link the contents of the Universe to its expansion, using the FRW metric. This relation is known as the *Friedmann Equation*.
- In General Relativity, this is done by solving *Einstein's equation of gravitation*:

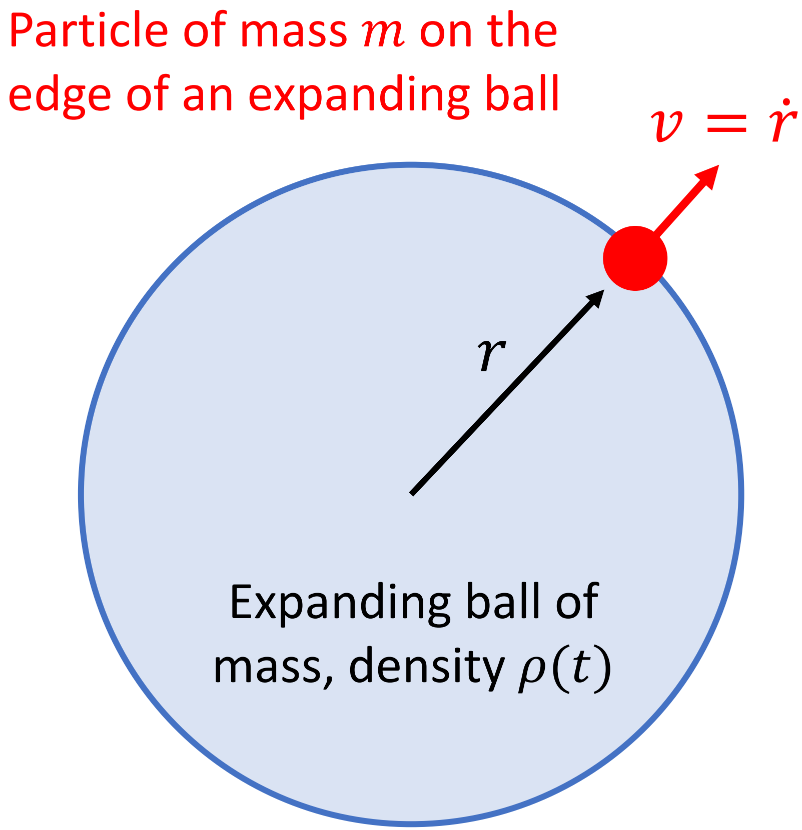
$G_{\mu\nu}$ is the Einstein tensor, which is a function of the space-time metric

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$T_{\mu\nu}$ is the energy-momentum tensor, which depends on the contents of space-time

The Friedmann equation

- We'll derive the solution to this equation in a *Newtonian setting*, which works remarkably well!



Conservation of energy for the particle:

$$\frac{1}{2} m\dot{r}^2 - \frac{GM(<r)m}{r} = E$$

Writing $M(<r) = \frac{4}{3}\pi r^3 \rho(t)$ and using comoving coordinates $r(t) = a(t)\, r_0$, this equation becomes,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\, \rho(t)}{3} - \frac{C}{a^2}$$

where $C = -2E/m{r_0}^2$ is a constant

The Friedmann equation

- This relation is remarkably close to the *full GR solution* containing curvature (K) and dark energy (Λ), which is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Expansion rate (where $\dot{a}/a$ is the Hubble parameter)

Effect of matter

Effect of curvature

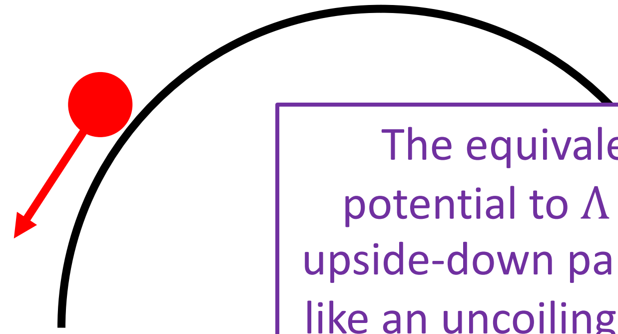
Effect of cosmological constant

- The Friedmann Equation is the **most important formula in cosmology**, since it links the expansion rate to the contents

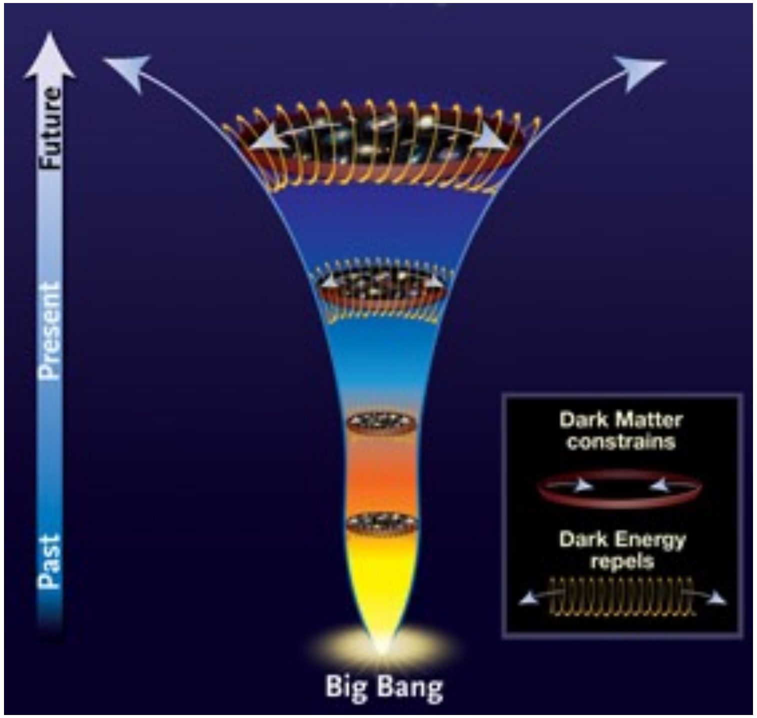
- *Note that curvature (K) is determined by the energy content [$\rho(t)$, Λ] so these terms cannot be varied completely independently*

The Friedmann equation

- It’s interesting to check how the Λ term appears in the Newtonian analysis
- Describing the Λ term as a “potential energy”, we can calculate it looks like an upside-down harmonic oscillator with $V(r) = -\frac{1}{6}\Lambda r^2$!
- Hence the Λ term is pushing the particle to accelerate away, like an uncoiling spring!



The equivalent potential to Λ is an upside-down parabola, like an uncoiling spring



The Friedmann equation

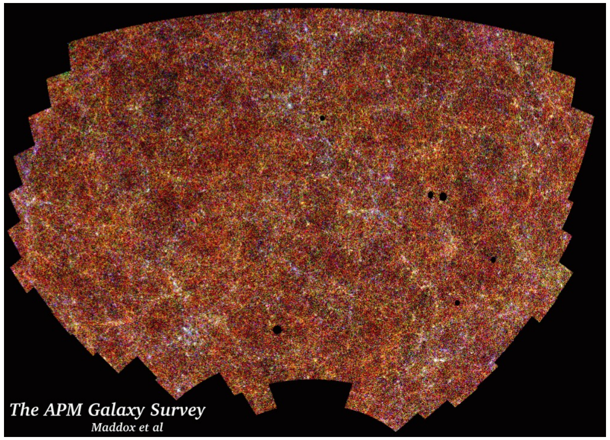
- $\rho(t)$ could be matter and/or radiation, varying with $a(t)$ as:

- *Matter:* $\rho_m(t) = \rho_0/a^3$
  (since the mass density dilutes as the Universe expands)

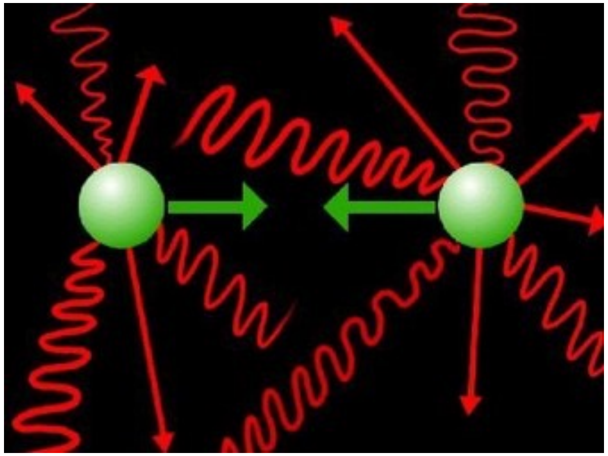
- *Radiation:* $\rho_r(t) = \rho_0/a^4$
  (since the energy density dilutes due to both expansion and redshifting of photons)

- *Dark energy:* $\rho_\Lambda(t) = \text{constant}$
  (since the energy density per unit volume is constant)

Matter: the smoothed-out distribution of galaxies

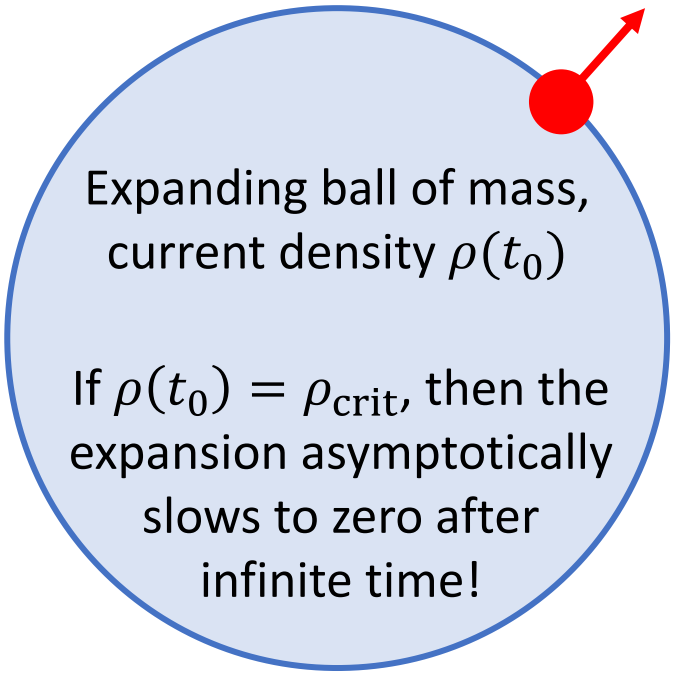


Radiation: the CMB photons filling the Universe



The critical density

- Consider a flat matter-dominated Universe:



- The expansion equation:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\, \rho(t)}{3} \propto E$$

- $E > 0$: expansion continues forever
- $E < 0$: expansion slows down and recollapses after a certain time
- $E = 0$: expansion gradually slows and stops after infinite time

- The *critical density*, $\rho_{\text{crit}} = 3{H_0}^2/8\pi G$, is the matter density today (when $\frac{\dot{a}}{a} = H_0$) of this final case

Density parameters

- Using the critical density, we can define new density parameters $\Omega = \frac{\rho}{\rho_{crit}}$. For the various components:

Matter	Radiation	Curvature	Dark energy
$\Omega_m = \frac{8\pi G \rho_{m,0}}{3{H_0}^2}$	$\Omega_r = \frac{8\pi G \rho_{r,0}}{3{H_0}^2}$	$\Omega_K = -\frac{Kc^2}{{H_0}^2}$	$\Omega_\Lambda = \frac{\Lambda c^2}{3{H_0}^2}$

- Using $\rho_m(t) = \rho_{m,0}/a^3$ and $\rho_r(t) = \rho_{r,0}/a^4$, the Friedmann equation then becomes:

$$\frac{1}{{H_0}^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda$$

- Evaluating today: $\Omega_r + \Omega_m + \Omega_\Lambda + \Omega_K = 1$ (this constraint shows that curvature is determined by the other parameters)

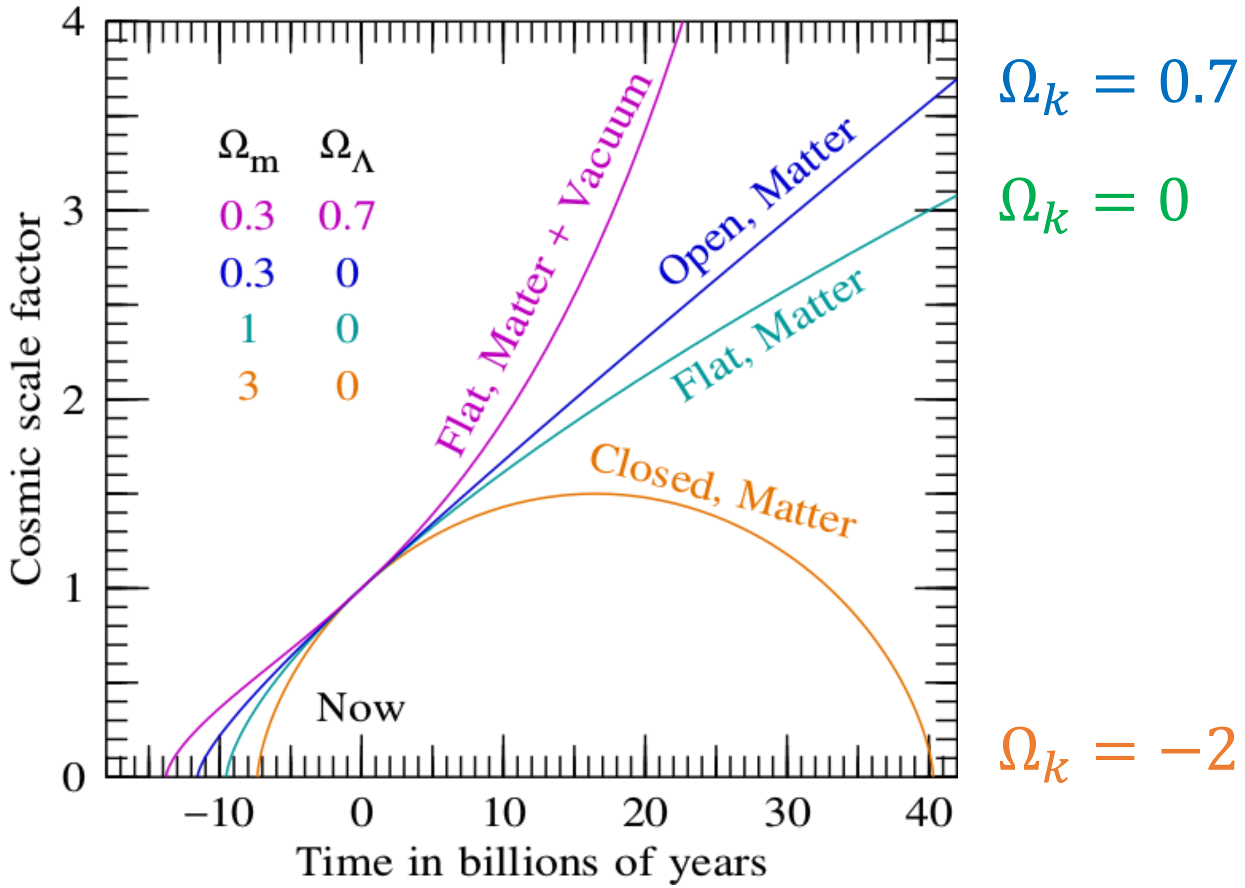
Solutions to the Friedmann equation

- Let's consider how the Universe would expand if it only consisted of **only one** of these components (we'll solve some of these cases in class…)

Universe	$\boldsymbol{\Omega_m}$	$\boldsymbol{\Omega_r}$	$\boldsymbol{\Omega_K}$	$\boldsymbol{\Omega_\Lambda}$	$\boldsymbol{a(t)}$	**Name**
Matter-dominated	1	0	0	0	$\propto t^{2/3}$	Einstein – de Sitter universe
Radiation-dominated	0	1	0	0	$\propto t^{1/2}$	
Empty	0	0	1	0	$\propto t$	Milne universe
Dark-energy dominated	0	0	0	1	$\propto e^{Ct}$	De Sitter space

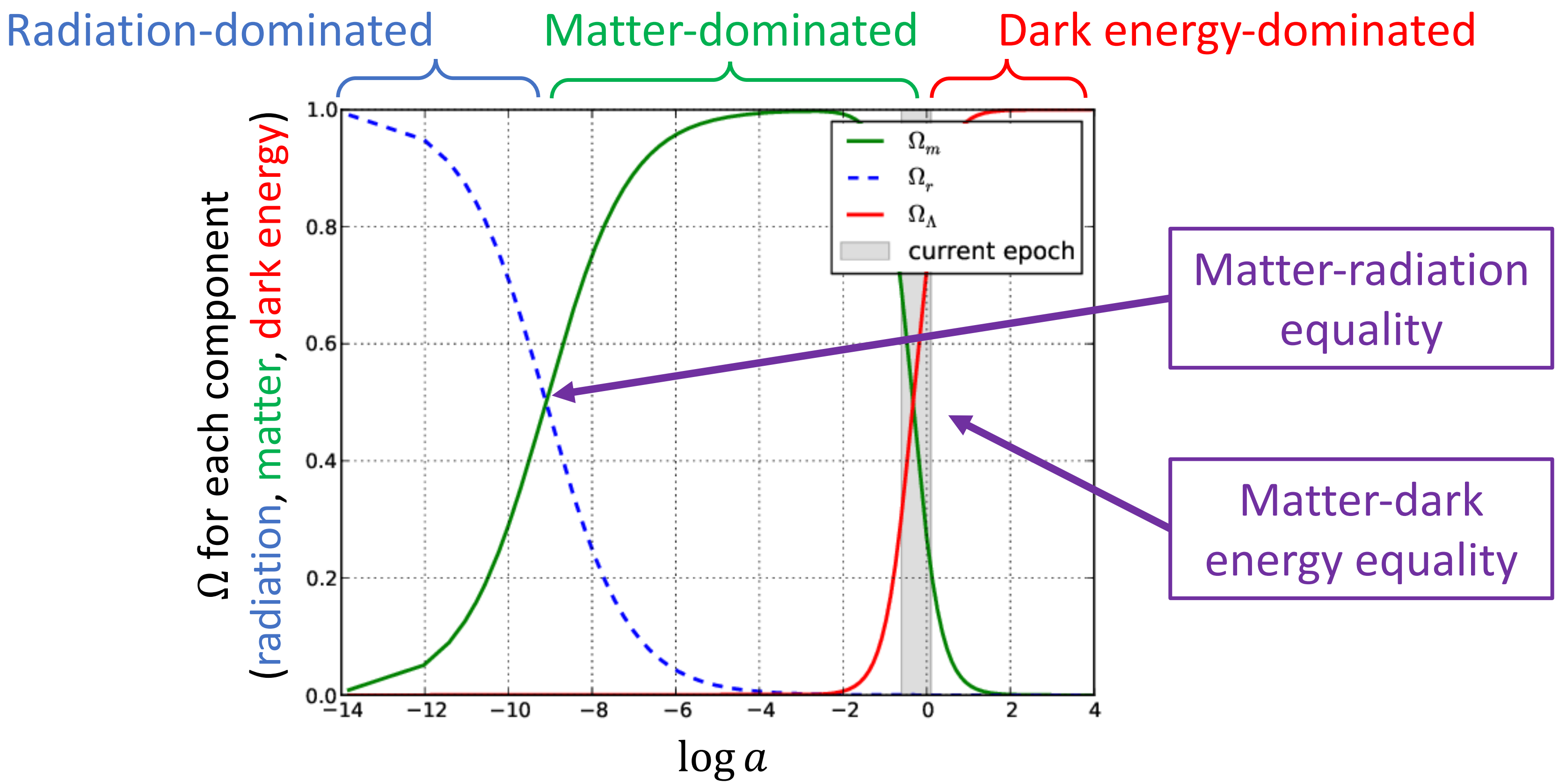
Solutions to the Friedmann equation

- We can also solve the Friedmann equation numerically for combinations of Ω's !



Solutions to the Friedmann equation

- The different scalings of the energy density in the components ($\rho_r \propto 1/a^4$, $\rho_m \propto 1/a^3$, $\rho_\Lambda = \text{const}$) imply that the Universe goes through *phases of expansion*:



Solutions to the Friedmann equation

- We can use the Friedmann equation to calculate the distance-redshift and time-redshift relations

- Here's an example for a flat Universe with $\Omega_m = 1$

- Looking back at Week 2, light travels with $ds = 0$, so from the metric we can write: $c\,dt = -a(t)\,dr$

- Hence, the coordinate distance corresponding to a given travel time is: $r = \int dr = c \int_t^{t_0} \frac{dt}{a(t)}$

- We can replace in the integral: $\frac{dt}{a(t)} = \frac{da}{a\,(\frac{da}{dt})} = \frac{da}{a^2\,H(a)}$, since $H(a)$ is given by the Friedmann equation

- For $\Omega_m = 1$, $H(a) = H_0\,a^{-3/2}$

- Hence $r = c \int_a^1 \frac{da}{a^2\,H(a)} = \frac{c}{H_0} \int_a^1 \frac{da}{a^{1/2}} = \frac{c}{H_0}\left[2a^{1/2}\right]_a^1 \rightarrow r = \frac{2c}{H_0}(1 - \sqrt{a})$

- For $\Omega_m = 1$, $a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}$

- Hence, $t = \frac{2}{3H_0} a^{3/2}$

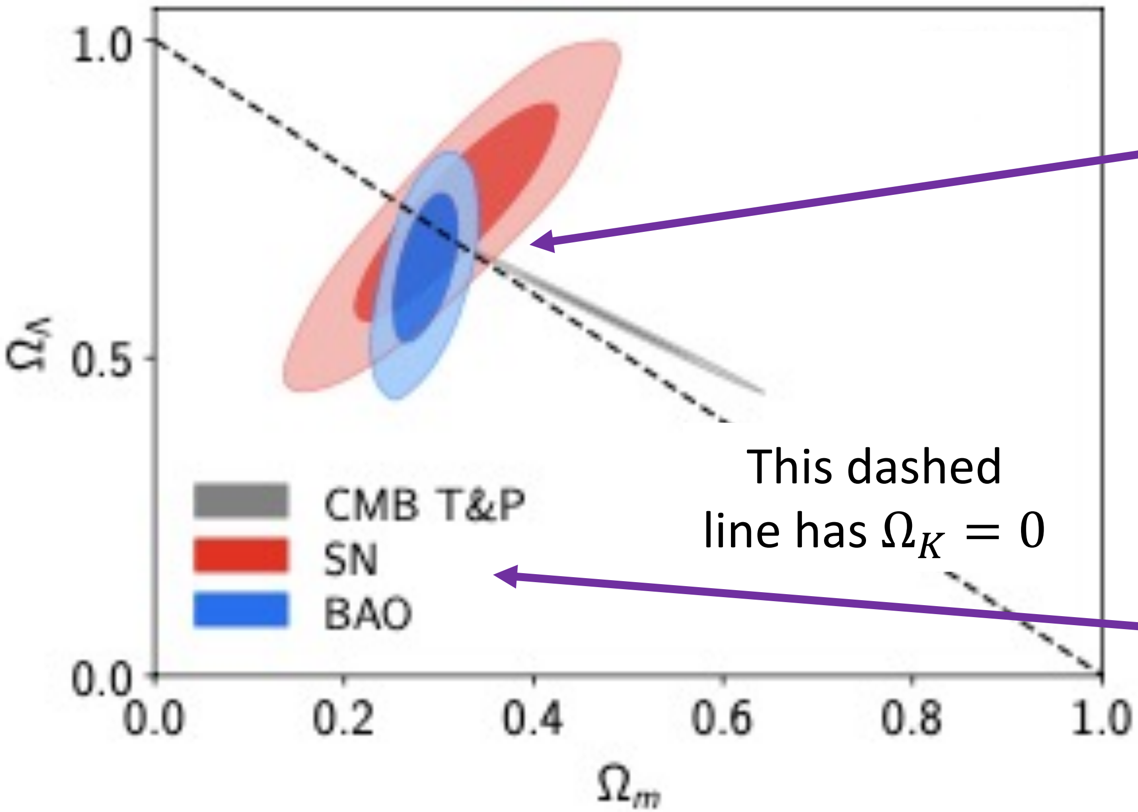
- (We can use $a = \frac{1}{1+z}$ in these equations)

Cosmological parameters

- What does cosmological data currently tell us about these density parameters?

We seem to live in a Universe with $\Omega_m \approx 0.3$, $\Omega_\Lambda \approx 0.7$ that has entered a phase of accelerating expansion!

We call it the **"ΛCDM Universe"**



This dashed line has $\Omega_K = 0$

We'll learn about these different types of observations in Week 6!

Energy conservation equation

- As the Universe expands, its energy density dilutes

It expands by dV in time dt

A component of the Universe with density ρ that exerts pressure P, occupies volume V and has internal energy E

From thermodynamics, this expansion causes the gas to lose internal energy,

$$dE = -P\,dV \rightarrow \frac{dE}{dt} = -P\frac{dV}{dt}$$

Substituting in this relation:

- $E(t) = mc^2 = \rho(t)\,V(t)\,c^2$
- $V(t) = V_0\,a(t)^3$ (for expansion)

we find, using $H = \dot{a}/a$,

$$\frac{d\rho}{dt} + 3H\left(\rho + \frac{P}{c^2}\right) = 0$$

Equation of state

- The relation between the pressure and density of a substance is called its *equation of state*, w:

$$P = w\rho c^2$$

- This allows us to unify the different components!
- Substituting in the equation on the previous slide:

$$\rho(t) = \rho_0 \, a^{-3(1+w)}$$

- Agrees with earlier expressions!

For **matter**:

$$w = 0$$

Matter is pressureless because it moves very slowly!

For **radiation**:

$$w = \frac{1}{3}$$

You'll find this relation in the kinetic theory of the gases!

For **dark energy**:

$$w = -1$$

Negative pressure propels the expansion to accelerate!

Acceleration equation

- Differentiating the Friedmann equation with respect to time and combining it with the energy equation, we find:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) = -\frac{4\pi G\rho}{3}(1 + 3w)$$

- This result shows that a component of the Universe must satisfy $w < -\frac{1}{3}$ for accelerating expansion ($\ddot{a} > 0$)

- The combination of the Friedmann equation and energy conservation equations is a closed set that allows us to determine the expansion history of any model

Key take-aways

- The main constituents in the standard model of cosmology are **radiation**, **matter**, **dark energy** and **curvature**
- The **Friedmann equation** links the expansion of the Universe to its matter-energy content (described by Ω parameters)
- Current observational data favours a **"ΛCDM model"** where $\Omega_m \approx 0.3, \Omega_\Lambda \approx 0.7, \Omega_K \approx 0$
- In this model, the Universe has **different phases of domination** by radiation, then matter, then dark energy
- All components can be described in terms of an **equation of state**, w, which links the pressure and energy density