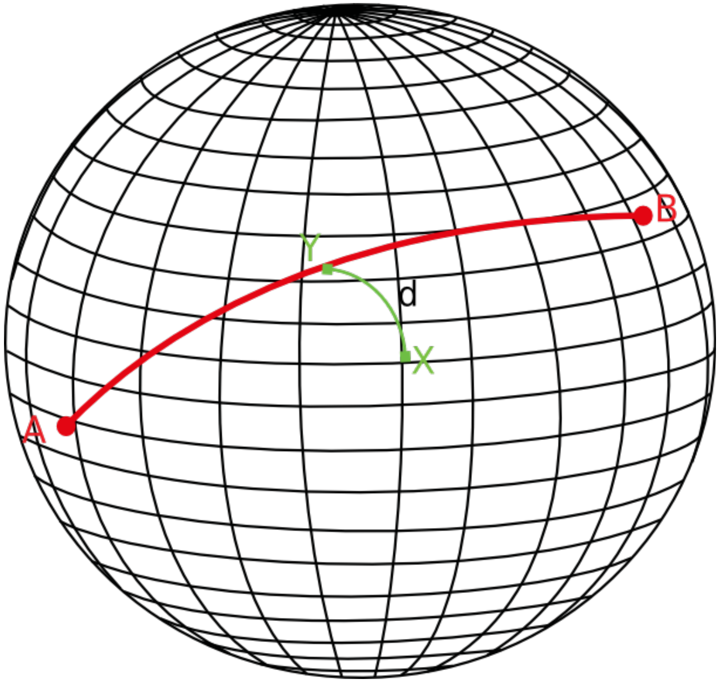
Honours Cosmology Week 2:

Measuring the Universe

This week we'll explore the space-time metric of the Universe, and how it may be used to describe cosmic distances and light travel



Measuring the Universe

At the end of this week you should be able to …

- … understand how **metrics** may be used to compute distances in a given coordinate system and topology
- … understand the form of the **Friedmann-Robertson-Walker (FRW) metric** of a homogeneous & isotropic Universe
- … use this metric to write down the **equation for light propagation** in an expanding Universe
- … describe different definitions of **cosmological distance**
- … determine the **age of the Universe** given the scale factor

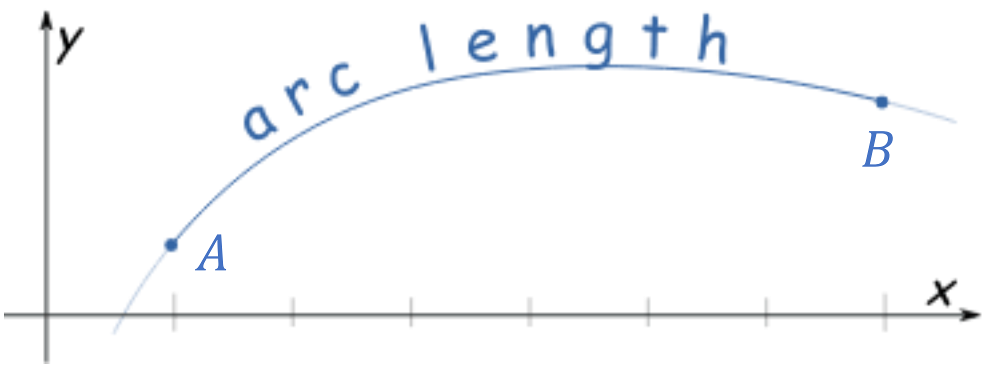
Measuring the Universe

- This week we'll learn about the equations for **mapping out the Universe** – that is, how to relate distances, redshifts, angles and light travel time

Introducing metrics

- Let's first consider how to measure distances in a 2D Cartesian space with coordinates (x, y)

Consider a curve between points A and B:



We parameterize the curve as $[x(\lambda), y(\lambda)]$

- We break the arc into small pieces: $ds^2 = dx^2 + dy^2$
- Integrating along the curve: $s = \int_A^B \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\, d\lambda$
- The relation between the line element ds and coordinate intervals (dx, dy) is called the **metric** of the space

Introducing metrics

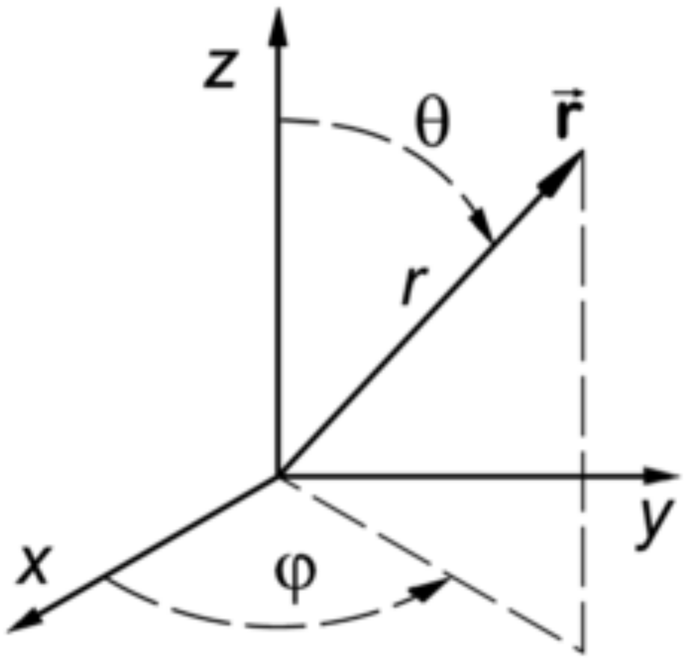
- The metric depends on the *chosen co-ordinate system*

For 3D Cartesian coordinates, we have:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Describing *exactly the same space* with spherical polar co-ordinates:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$



For example, to calculate the distance to circumnavigate the world from Melbourne on a line of constant latitude θ_0: $dr = 0$ and $d\theta = 0$, so

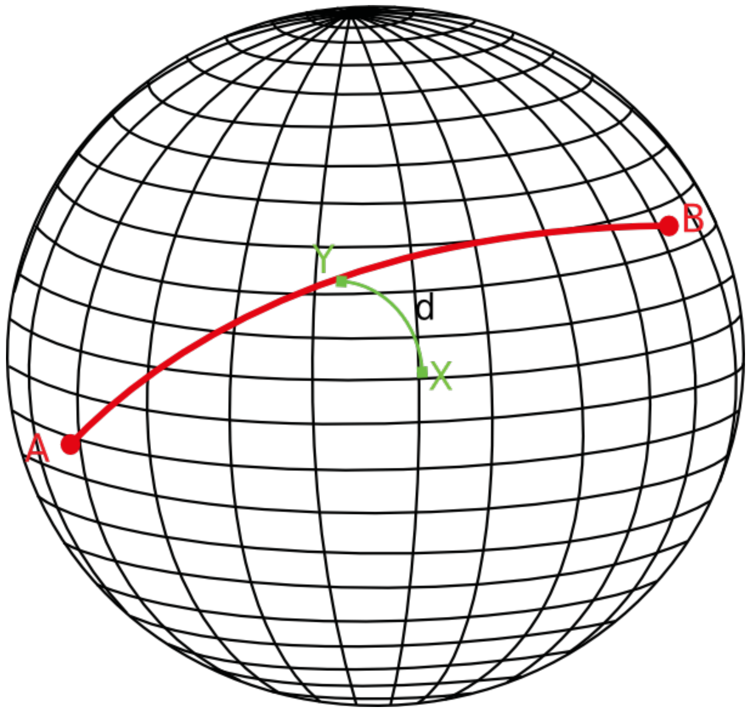
$$s = \int ds = r_{\text{Earth}} \sin\theta_0 \int_0^{2\pi} d\phi = 2\pi\, r_{\text{Earth}} \sin\theta_0$$

Introducing metrics

- The metric also depends on the *intrinsic topology of the space*

- To describe the constant curvature surface of a unit sphere, we could use spherical polar co-ordinates with $r = 1$:

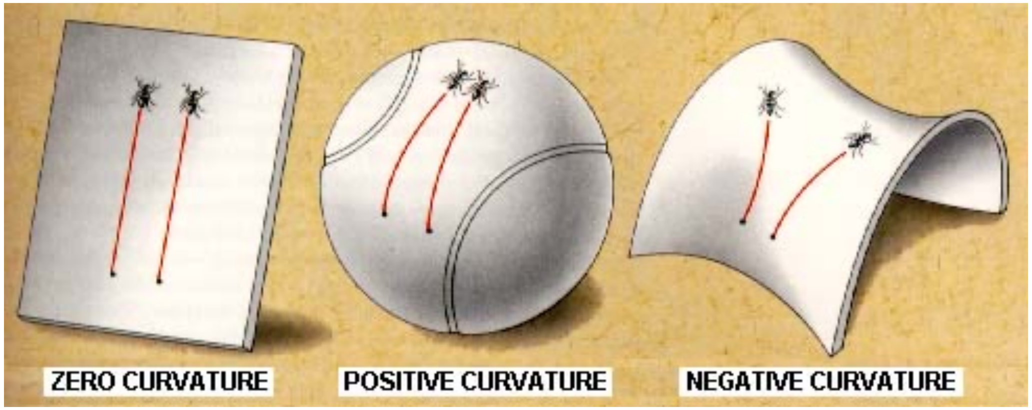
$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

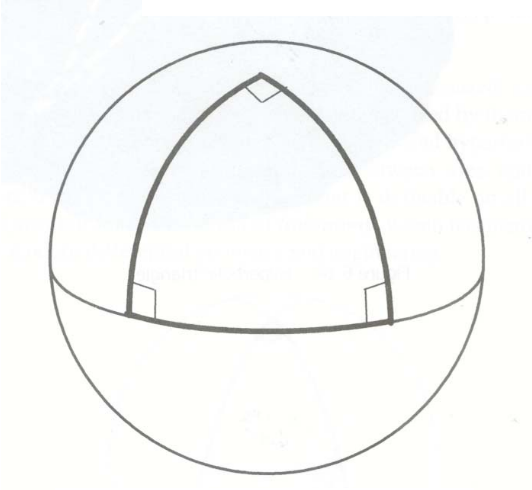


- A 2D flat space would be described by $ds^2 = d\theta^2 + d\phi^2$: in this case, *the metrics are different but so are the intrinsic topologies (curved space vs. flat space)*

Curved spaces

- How do curved spaces differ from flat spaces?





- In a curved space: parallel lines converge or diverge
- The circumference of a circle $\neq 2\pi r$
- The sum of the angles of a triangle $\neq 180°$

- *The curvature of a space can be determined by local observers without needing to see the space from outside*

Curved spaces

- How do we generalise the metric of a constant curvature space from 2D, as above, to 3D?

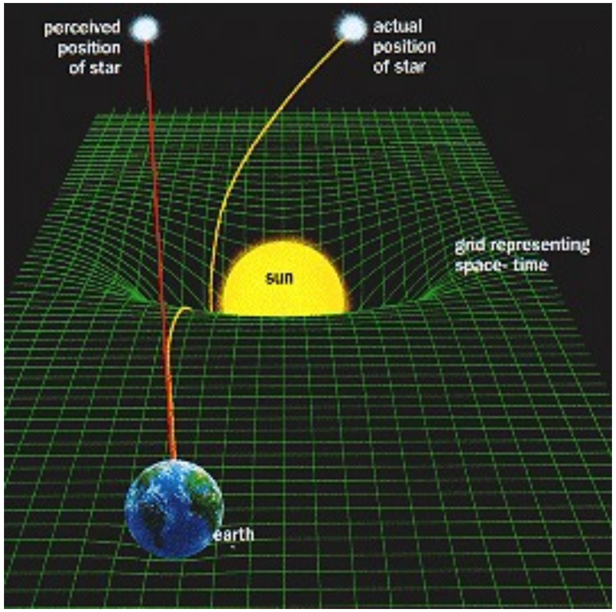
- A constant curvature 2D surface embedded in a 3D Euclidean space satisfies the equation: $x^2 + y^2 + z^2 = 1/K$ (K = curvature parameter)
- For a constant curvature 3D surface embedded in a 4D Euclidean space: $x^2 + y^2 + z^2 + w^2 = 1/K$ (adding the extra co-ordinate w)
- In the 4D Euclidean space: $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$
- We now transform (x, y, z) to spherical polar co-ordinates (r, θ, ϕ)
- From above: $w^2 = 1/K - r^2$, hence $dw^2 = \frac{r^2\, dr^2}{1/K - r^2}$
- Putting it all together: $ds^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

Introducing General Relativity

- The Universe is described by *Einstein's theory of General Relativity*, summarised in two phrases:

Space-time tells matter how to move

Matter tells space-time how to curve



- We won't study GR in detail in this module, but we'll use a key aspect of it: the space-time metric

The metric of the Universe

- In relativity, the metric describes separations in "space-time" rather than just "space"

- We already know the space-time interval in special relativity (invariant for observers in all inertial frames):

$$ds^2 = -c^2dt^2 + dx^2$$

- We obtain the metric of the Universe by replacing dx with the line element for comoving coordinates of a constant curvature space expanding with the Universe

- We must preserve constant curvature to satisfy the conditions that the Universe is homogeneous and isotropic

The metric of the Universe

- Combining these results, we obtain the space-time metric of the expanding Universe:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]$$

Same term as in special relativity, where t is the time measured by fundamental observers

Multiplied by the cosmic scale factor to convert comoving coordinates to physical separation

Line element of a space with constant curvature K in comoving coordinates (r, θ, ϕ)

- This form is called the *Friedmann-Robertson-Walker (FRW) metric*, after its discoverers

Light travel through the Universe

- We can use this space-time metric to study a *light ray travelling through the Universe* on a radial path from a galaxy at coordinate r_{em} to $r = 0$

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- Remember from special relativity, light travels between two points of space-time such that $ds = 0$
- Since $d\theta = d\phi = 0$, we find: $c\,dt = -\frac{a(t)\,dr}{\sqrt{1-Kr^2}}$

Choosing the negative solution of the square root, since r is decreasing

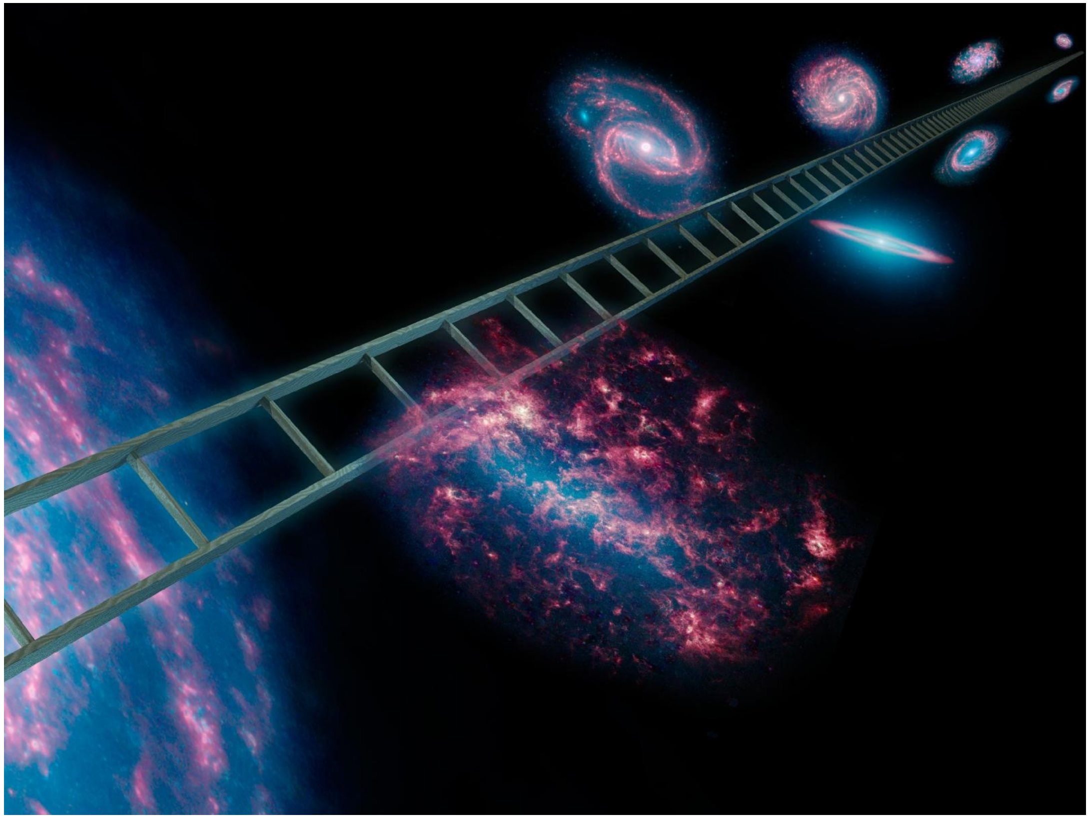
- Integrating along the path, $\int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_{\mathrm{em}}} \frac{dr}{\sqrt{1-Kr^2}}$

Light travel through the Universe

- For a second photon emitted slightly later and following the first, we would likewise find: $\int_{t_{\rm em}+\delta t_{\rm em}}^{t_{\rm obs}+\delta t_{\rm obs}} \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_{\rm em}} \frac{dr}{\sqrt{1-Kr^2}}$
- Since the right-hand side of the previous two equations is the same, we can conclude: $\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} = \int_{t_{\rm em}+\delta t_{\rm em}}^{t_{\rm obs}+\delta t_{\rm obs}} \frac{dt}{a(t)}$
- From which we deduce the key result: $\frac{\delta t_{\rm obs}}{a(t_{\rm obs})} = \frac{\delta t_{\rm em}}{a(t_{\rm em})}$
- The time between the photons is inversely proportional to the frequency ω of the light, $\delta t \propto 1/\omega \propto$ the wavelength λ
- Hence the light is redshifted: $\lambda_{\rm obs} = \frac{a(t_{\rm obs})}{a(t_{\rm em})}\lambda_{\rm em} = (1+z)\,\lambda_{\rm em}$

Distances in expanding space

- How do we describe *distances in the expanding Universe*, from $r = 0$ to a galaxy at coordinate r?



Distances in expanding space

- The first approach we might consider is evaluating the physical distance (or "proper distance") between the origin and the galaxy at a chosen time. From the metric:

$$dD = \frac{a(t)}{\sqrt{1-Kr^2}}\,dr \quad \text{(since } d\theta = d\phi = dt = 0\text{)}$$

- Integrating between 0 and r we find results depending on K:

$$D = a(t)\sin^{-1}\left(r\sqrt{K}\right)/\sqrt{K} \qquad K > 0$$

$$D = a(t)\,r \qquad K = 0$$

$$D = a(t)\,\sinh^{-1}\left(r\sqrt{-K}\right)/\sqrt{-K} \qquad K < 0$$

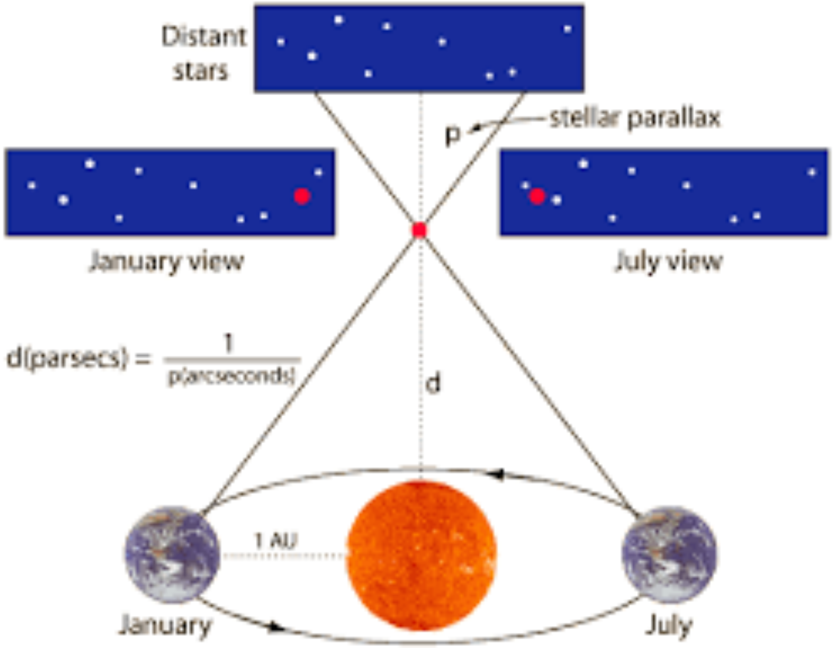
Distances in expanding space

- However, the physical distance is not a very useful quantity in observational cosmology, since *we can't in practice arrange for such a measurement to be made!*

- How can we actually measure distances?

- A nice method would be to use the shift in angular position of an object due to the orbit of the Earth, called a *parallax distance*

- However, for cosmological distances the angular shift is too small, so this also doesn't work!



Distances in expanding space

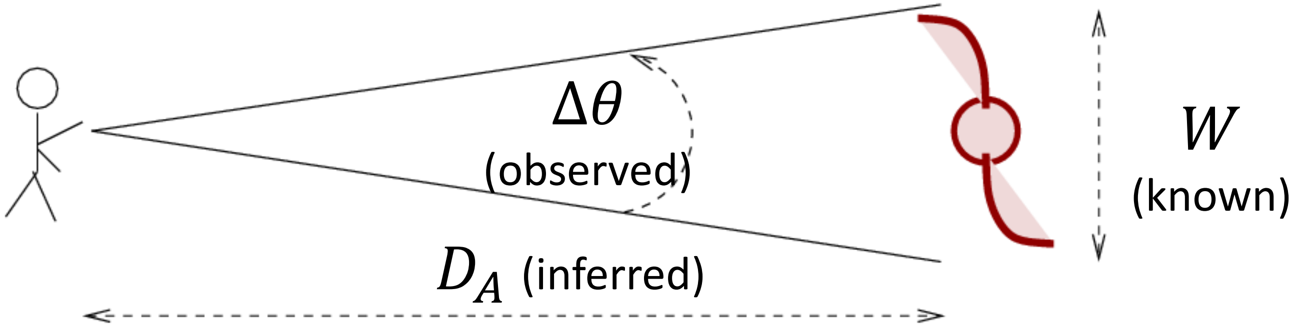
- More useful distance measures are based on the observed *size* or *brightness* of distant objects

Distance measurements based on the size or separation of distant objects produce an **angular diameter distance**

Distance measurements based on the brightness of distant objects produce a **luminosity distance**

Angular diameter distance

- Angular diameter distance D_A is defined using the *observed angular size $\Delta\theta$ of an object of known physical width W*

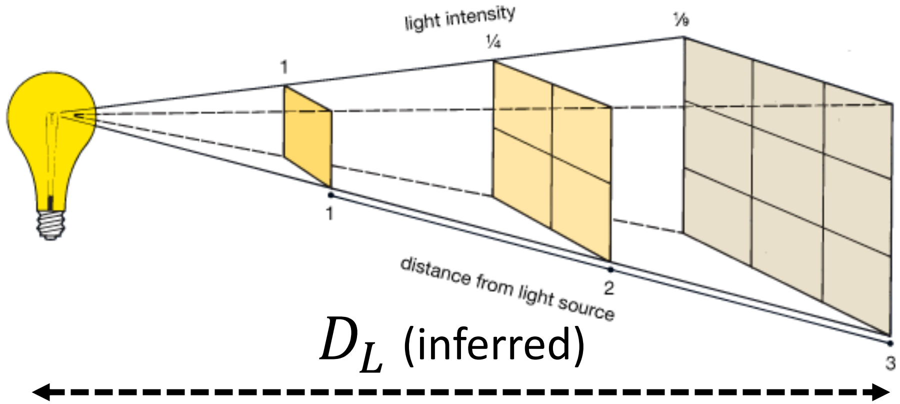


- The angular diameter distance is defined as: $D_A = W/\Delta\theta$
- From the metric: $W = a(t)\, r\, \Delta\theta$, where r = object coordinate
- Hence: $D_A = a(t_{\mathrm{em}})\, r = \frac{r}{1+z}$ (t_{em} = emission time)
- Objects of known physical width W are called *standard rulers*

Luminosity distance

- Luminosity distance D_L is defined using the *flux f received from an object of known luminosity L*

Luminosity L
(known)



Flux f
(observed)

D_L (inferred)

- The usual "inverse square law" would give us: $f = \frac{L}{4\pi {D_L}^2}$

- This motivates our definition: $D_L = \sqrt{\frac{L}{4\pi f}}$

Luminosity distance

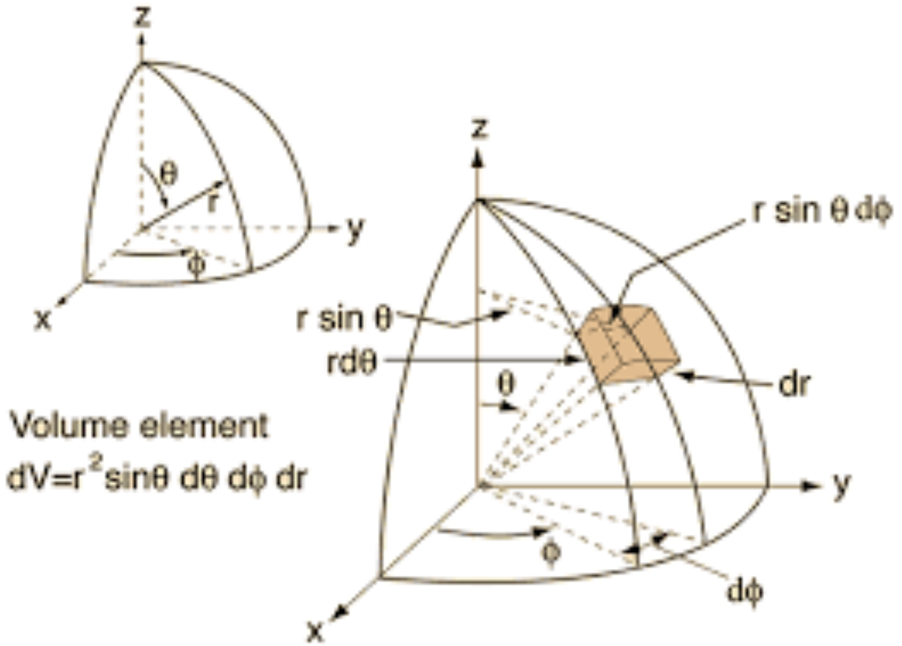
- To relate the luminosity distance to the object's coordinate r, we have to take two cosmological effects into account:
  1. Photons lose energy as their wavelength increases owing to the expansion of the Universe [effect $\propto a(t_{\mathrm{em}})$]
  2. The time interval between arriving photons also increases (see the calculation on slide 13) [effect $\propto a(t_{\mathrm{em}})$]
- Hence, the flux of energy received is: $f = \frac{L\,a(t_{\mathrm{em}})^2}{4\pi r^2}$
- Hence: $D_L = \sqrt{\frac{L}{4\pi f}} = \frac{r}{a(t_{\mathrm{em}})} = r\,(1+z)$
- Objects of known luminosity L are called *standard candles*

Volumes

- The volume element in comoving space between coordinates r and $r + dr$ can also be deduced from the metric:

$$dV = 4\pi r^2 \frac{dr}{\sqrt{1 - Kr^2}}$$

(this formula is for the full sky, for part of the sky we take a fraction)



It's similar to the standard result for spherical polar coordinates, modified for the effect of curvature

Age of the Universe

- The age of the Universe is the coordinate time t that elapses between $a = 0$ and $a = 1$

- As an estimate, we can extrapolate today's expansion rate $\frac{da}{dt} = H_0$:

$$a(t) = 1 + H_0(t - t_0)$$

- This leads to an approximate age $\approx \frac{1}{H_0} = \text{Hubble time} = 14\ \text{Gyr}$ if $H_0 = 70\ \text{km s}^{-1}\ \text{Mpc}^{-1}$

- More precisely we would have an integral: $t_{\text{age}} = \int dt = \int_0^1 \frac{da}{a\,H(a)}$

Age of the Universe

- Does an age of ≈ 14 billion years make sense?

Geological evidence suggests the Earth is ≈ 4.5 billion years old

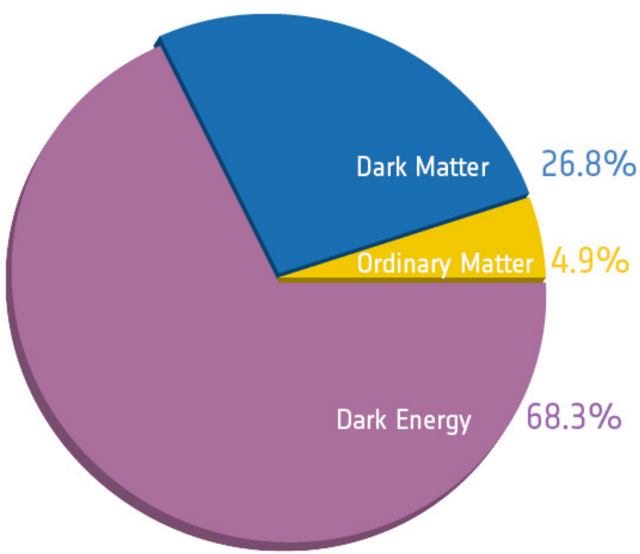
Radioactive dating shows meteorites are ≈ 4.5 billion years old

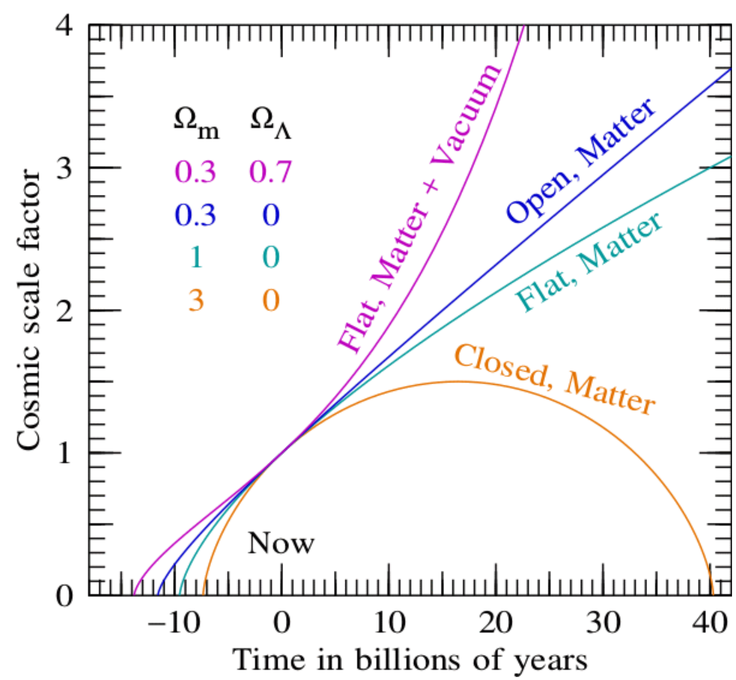
Some white dwarf stars are known to be ≈ 12 billion years old

Globular clusters can also be dated as ≈ 12 billion years old

Next steps …

- To go further in our calculations, we'll need to know how the scale factor $a(t)$ depends on time
- This depends on the contents of the Universe, which we'll study next week!





Key take-aways

- **Metrics** of a space allow us to relate distance elements to changes in the coordinates of the space
- The Universe is described by combining **General Relativity** with the assumptions of homogeneity and isotropy
- The result is the **Friedmann-Robertson-Walker space-time metric** of the expanding Universe, which is written in terms of a constant curvature K and scale factor $a(t)$
- We can measure distances in cosmology using the **luminosity distance** or **angular diameter distance**
- The **age of the Universe** may be deduced as the time elapsed between $a = 0$ and $a = 1$