Cosmology Assignment 1

Weeks 1 & 2: The Expanding Universe & Measuring the Universe

Q1) Suppose that you are a two-dimensional being, living on a sphere of radius R. Show that if you draw a circle of radius r around your location, the circle's circumference will be

$$C = 2\pi R \sin(r/R)$$

The radius of the Earth is $R = 6371$ km. If you can measure distances with an accuracy of ± 1 m, what radius of circle would you have to draw on the Earth's surface to convince yourself that the Earth is curved rather than flat?

Q2) An early hypothesis used to explain Hubble's law is the "tired light" hypothesis. The tired light hypothesis states that the Universe is not expanding, but that photons simply lose energy as they move through space (by some unexplained means!), with energy loss per unit distance being given by:

$$\frac{dE}{dr} = -K\,E$$

where K is a constant. Show that this hypothesis reproduces Hubble's law at low redshifts, and find the value of K which yields a Hubble constant $H_0 = 70\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$. How might the tired light hypothesis be disproven?

Q3) Suppose that our Universe has a space-time metric given by the following expression:

$$ds^2 = -c^2 dt^2 + \left(\frac{t}{t_0}\right)^{4/3} \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

in terms of time co-ordinate t and spatial spherical polar co-ordinates (r, θ, ϕ), where t_0 is a constant.

a) What is the global curvature of this Universe? Explain why the scale factor as a function of time is given by $a(t) = (t/t_0)^{2/3}$. What property of this Universe is given by the constant t_0?

b) Show that the Hubble parameter of this Universe as a function of redshift z is:

$$H(z) = \frac{2(1+z)^{3/2}}{3t_0}$$

The expansion rate of this Universe today (at redshift $z = 0$) is measured to be $H_0 = 70\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$. Observers today detect a light signal from an alien civilisation on a galaxy at redshift $z = 0.5$.

c) How long ago was this light signal emitted?

d) By setting $ds = 0$ in the space-time metric, show that the radial co-ordinate of the galaxy emitting the light is given by

$$r = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

e) The light signal arrives at the $z = 0$ observers with a flux of 10^{-17} J m^{-2} s^{-1}. Assuming the signal was emitted isotropically, what was the luminosity of the signal in units of watts (W)?

Q4) The redshift of a galaxy slowly changes with time as the Universe ages, in an effect called "redshift drift". Using the standard relations between redshift, scale factor and time,

$$1 + z = \frac{a_o}{a_e} = \frac{dt_o}{dt_e}$$

where z is the currently-observed redshift, a_o and a_e are scale factors at the time of observation and emission of light, and dt_o and dt_e are time intervals around the observation and emission times, derive the redshift drift formula:

$$\frac{dz}{dt_o} = (1 + z)\, H_0 - H(z)$$

in terms of the Hubble parameter.

An astronomer's spectrograph has a resolving power of $\frac{\lambda}{\Delta\lambda} = 10^6$. In a matter-dominated Universe with the critical density (as analysed in Q3), how long would the astronomer need to observe a source at $z = 1$ to directly detect the expansion of the Universe?