Class 8: Tensors

In this class we will explore how general co-ordinate transformations may be described by a tensor calculus using index notation, leading to a generalized notion of curvature

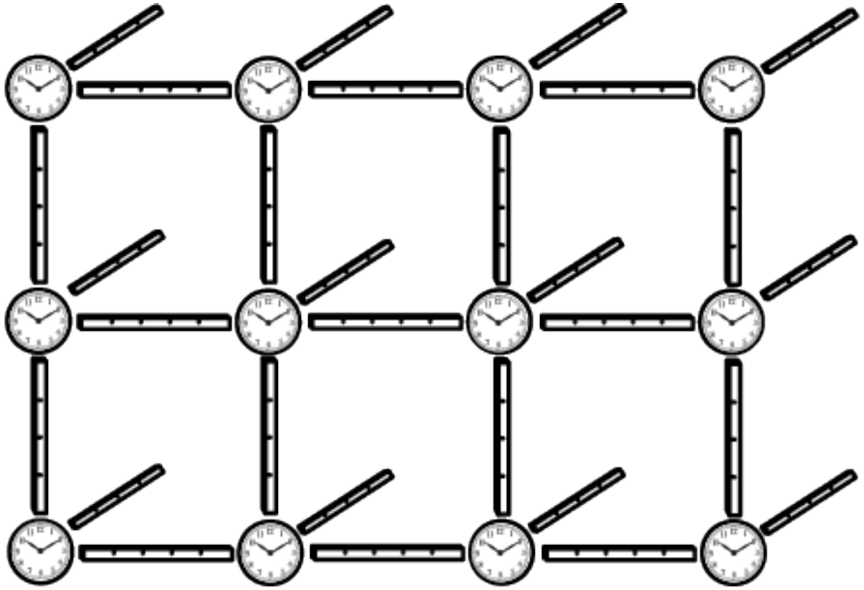
Class 8: Tensors

At the end of this session you should be able to …

- … understand how the Lorentz transformations may be replaced by **general co-ordinate transformations**
- ... describe why such transformations are **fundamental to formulating the laws of physics**
- … apply index notation to **manipulate general tensor objects**, such as by raising or lowering an index
- … describe how the notion of parallel-transport in a curved space leads to the generalized **Riemann curvature tensor**

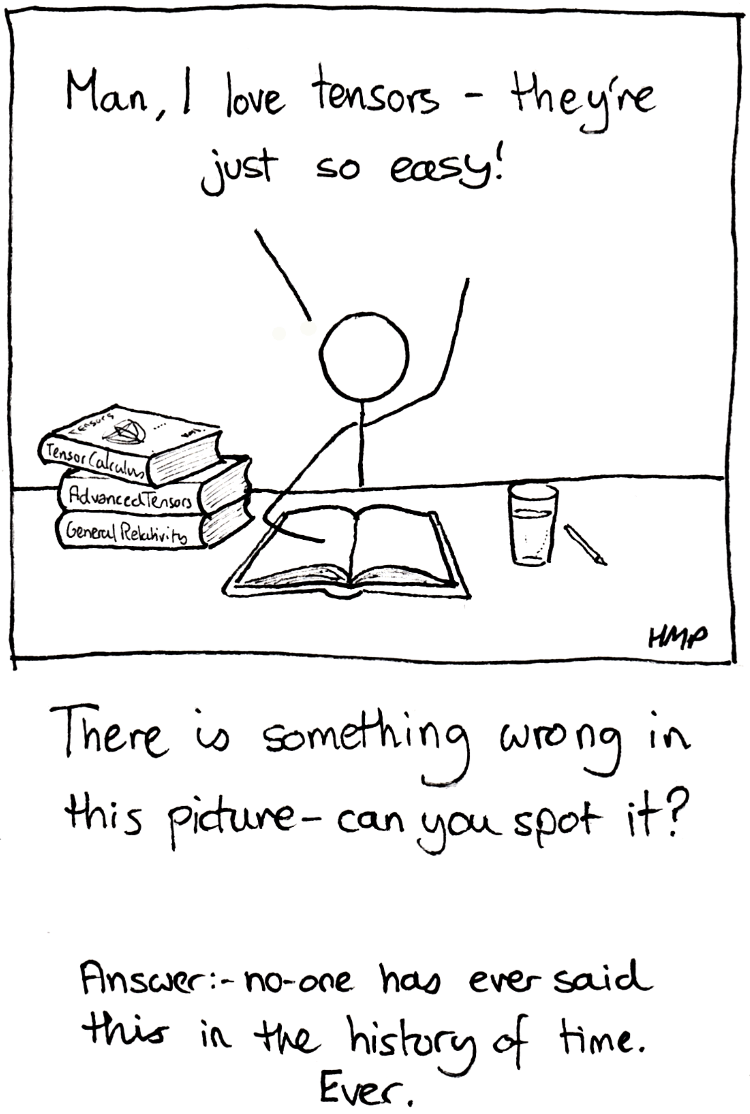
The laws of physics

- A fundamental idea of Relativity is ***all reference frames are equally suitable for the formulation of the laws of physics***
- A **reference frame** is a space-time observing system, such as the Earth's frame, or a freely-falling frame, or an inertial frame in SR



The laws of physics

- ***Physics does not depend on our choice of co-ordinate frame***
- An equation representing a physical law in co-ordinate frame x, such as $A^{\mu} = B^{\mu}$, must transform to a different frame x' such that $A'^{\mu} = B'^{\mu}$
- We need some powerful mathematics to ensure that this will happen – this is the **mathematics of tensor calculus**



https://comic.hmp.is.it/comic/tensor-calculus/

Special Relativity recap

- In Special Relativity we introduced the idea of a **4-vector**, a group of four quantities A^μ whose values in 2 inertial frames are related by the *Lorentz transformations*:

$$A'^{\mu} = L^{\mu}{}_{\nu} \, A^{\nu} \qquad L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -v\gamma/c & 0 & 0 \\ -v\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- It was convenient for us to define a **"down" 4-vector** A_μ:

$$A_\mu = \eta_{\mu\nu} \, A^\nu$$

$$A^\mu = \eta^{\mu\nu} \, A_\nu$$

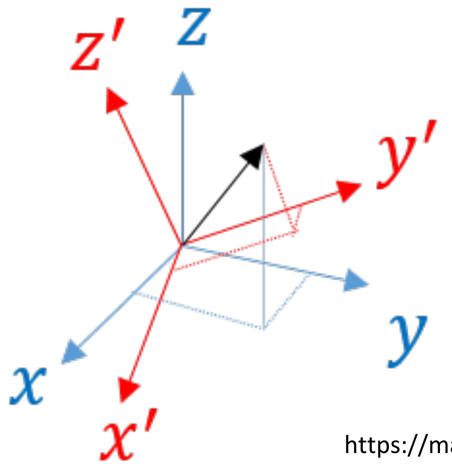
$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- This is because $A^\mu A_\mu$ is an invariant

General transformations

- The Lorentz transformation between inertial frames is a special case – we must develop mathematics to describe **an arbitrary transformation between 2 co-ordinate frames** (e.g. the Earth's frame, and a freely-falling frame)

- A co-ordinate transformation provides relations for some x' co-ordinates in terms of x co-ordinates, $x' = f(x)$



https://math.stackexchange.com/questions/1228106/how-can-i-transform-coordinate-systems-based-on-quaternion-data

General transformations

- We start by transforming simple differentials and gradients using the **chain rule**: $dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, dx^{\nu}$ and $\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\, \frac{\partial f}{\partial x^{\nu}}$

- Using this template …

- A "general up 4-vector" A^{μ} is an array whose values in the 2 frames are related by: $\boldsymbol{A'^{\mu}} = \boldsymbol{\frac{\partial x'^{\mu}}{\partial x^{\nu}}\, A^{\nu}}$ *(the Lorentz transformation is a special case of this with $x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$)*

- A "general down 4-vector" A_{μ} is an array whose values in the 2 frames are related by: $\boldsymbol{A'_{\mu}} = \boldsymbol{\frac{\partial x^{\nu}}{\partial x'^{\mu}}\, A_{\nu}}$

General transformations

- The general transformation of an “up” index to a “down” index uses the space-time metric:

$$A_\mu = g_{\mu\nu}\, A^\nu \qquad\qquad A^\mu = g^{\mu\nu}\, A_\nu$$

- $\boldsymbol{g^{\mu\nu}}$ **is the inverse matrix of $\boldsymbol{g_{\mu\nu}}$**, since applying both of these operations in turn to A^μ must restore the original quantity

- In an inertial or freely-falling frame, $g_{\mu\nu} = \eta_{\mu\nu}$, and we recover the previous rules for raising/lowering an index

Tensors

- *Some physical quantities are grouped into larger structures*

- More generally, a **tensor** $A^{\mu\nu}$ transforms between 2 frames as:

$$A'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} A^{\kappa\lambda} \qquad A'_{\mu\nu} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \frac{\partial x^{\lambda}}{\partial x'^{\nu}} A_{\kappa\lambda}$$

- We raise and lower indices using the metric, for example:

$$A_{\lambda}{}^{\nu} = g_{\lambda\mu} A^{\mu\nu}$$

$$A^{\mu}{}_{\lambda} = g_{\lambda\nu} A^{\mu\nu}$$

$$A_{\kappa\lambda} = g_{\kappa\mu}\, g_{\lambda\nu} A^{\mu\nu}$$

- We can generalize these relations to higher dimensions

Tensors

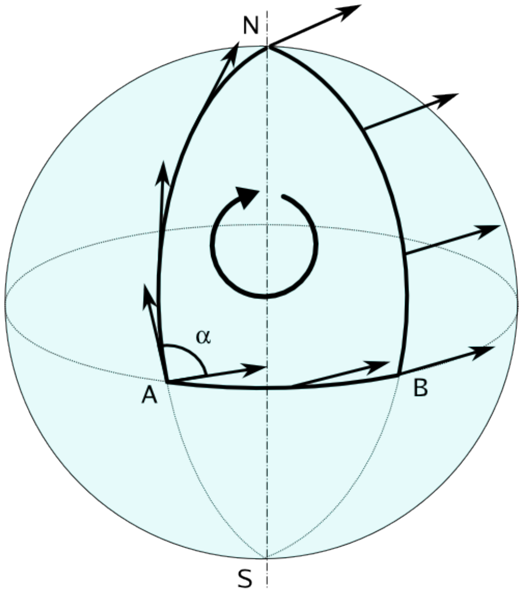
With this mathematical apparatus in hand we can derive a number of useful relations of tensor calculus:

- If $A^\mu = B^\mu$ then, in any other frame, $A'^\mu = B'^\mu$
- $A^\mu B^\nu$ is a tensor $C^{\mu\nu}$
- $A^\mu B_\mu$ is a scalar invariant in all frames
- $C^{\mu\nu} A_\nu$ is a 4-vector D^μ
- We can re-arrange summed indices, e.g. $A^\mu B_\mu = A_\mu B^\mu$

We have already met some tensors in the course, such as $g_{\mu\nu}$ and $T_{\mu\nu}$. We are about to meet some more!

General description of curvature

- *How can we describe the curvature of a region of space?*
- On the surface of a sphere, carry an arrow from the Equator to the Pole and back on a path $A \rightarrow N \rightarrow B \rightarrow A$ shown below
- Suppose we **parallel-transport** the arrow, meaning that its components are unchanged in a local Cartesian system

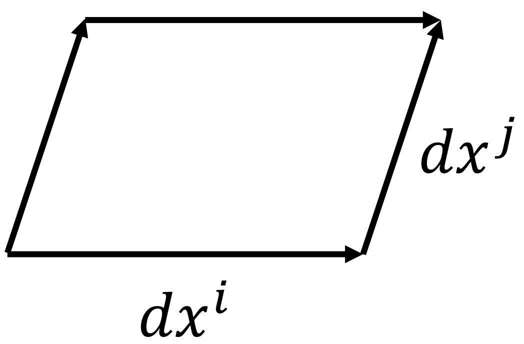


https://commons.wikimedia.org/wiki/File:Parallel_transport.png

Vectors parallel-transported around a closed path on a curved surface are rotated!

General description of curvature

- Suppose you are at a point x^μ in space-time
- Travel in direction i until your co-ordinate x^i is increased by a small amount dx^i, without changing the other co-ordinates
- Now travel in direction j until co-ordinate x^j is increased by dx^j, without changing the other co-ordinates
- Now move backwards by $-dx^i$
- Finally, move backwards by $-dx^j$
- You are, of course, back at x^μ!



The Riemann tensor

- Now travel the same route again, **parallel-transporting a vector** A^k which initially points in direction k

- Vectors parallel-transported around a closed path in a curved surface are rotated – let the change in each component be dA^l

- This thought experiment allows us to define the **Riemann tensor** $\boldsymbol{R^{\kappa}_{\lambda\mu\nu}}$, which provides a general measure of curvature:

$$dA^l = R^l_{kij}\, A^k\, dx^i\, dx^j$$

- The Riemann tensor may be expressed in terms of the Christoffel symbols: $R^{\kappa}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\kappa}_{\lambda\nu} - \partial_\nu \Gamma^{\kappa}_{\lambda\mu} + \Gamma^{\kappa}_{\mu\alpha}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\kappa}_{\nu\alpha}\Gamma^{\alpha}_{\lambda\mu}$

The Riemann tensor

- In an N-dimensional space, we have N^2 possible loops, and N final components of N initial vectors, i.e. N^4 components at each point, $= 256$ for $N = 4$!!

- It is not that bad, owing to symmetries (e.g., going backwards round the loop). Actually, the number of independent components is $\boldsymbol{N^2(N^2 - 1)/12}$

- So the curvature at each point is described by 1 number when $N = 2$ (on a 2D curved surface), and 20 numbers when $N = 4$