Class 5: Energy and Momentum

In this class we will study how the basic concepts of mechanics, energy and momentum, must be modified to be consistent with the postulates of relativity

Class 5: Energy and Momentum

At the end of this session you should be able to …

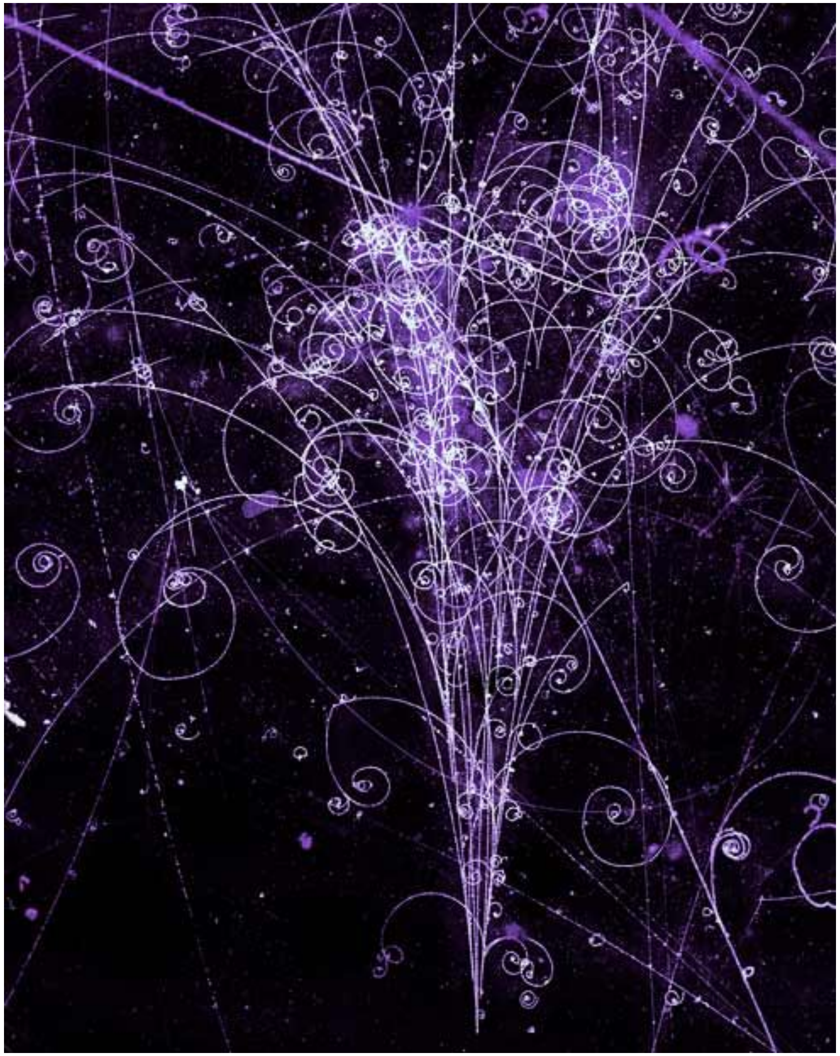
- … recall how the definitions of **momentum** and **energy** are modified in relativity
- … apply **conservation laws** of these quantities, to study relativistic particle collisions
- … be familiar with the concept of the **rest mass** of a particle, and how it represents an **equivalent energy**
- ... understand how to calculate the energy and momentum of **photons**, particles travelling at the speed of light

Relativistic mechanics

- Every day in the world's most energetic particle colliders, billions of particles smash together at relativistic speeds

- Protons in the Large Hadron Collider are moving with speed 0.99999999 c !!

- Do the normal laws of mechanics apply at these speeds?



https://www.pinterest.com.au/pin/542120873871190260/

Relativistic mechanics

- The most important aspect of mechanics is **conservation laws**
- They allow us to analyse the most complex interactions in terms of simple principles

https://www.flexptnj.com/the-ride-of-your-life/

http://www.alomabowlingcenters.com/boardwalk/bowl-a-strike-at-boardwalk-bowl/

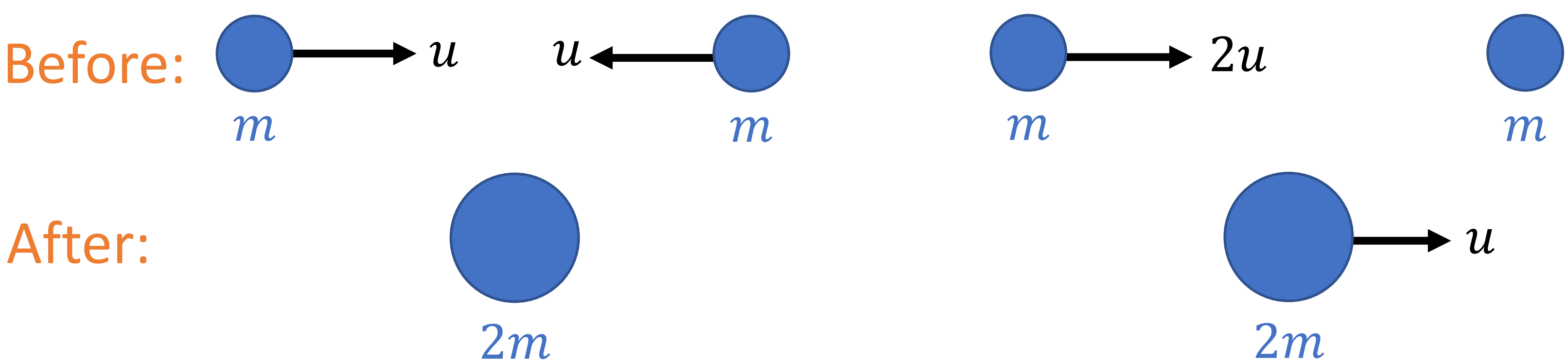
Modifying Newtonian mechanics

- To preserve conservation laws at high speeds requires us to **modify the definitions of energy and momentum**

- To see why, consider a simple collision in two frames:

Frame S'

Frame S [sitting with 2nd particle] (classical)

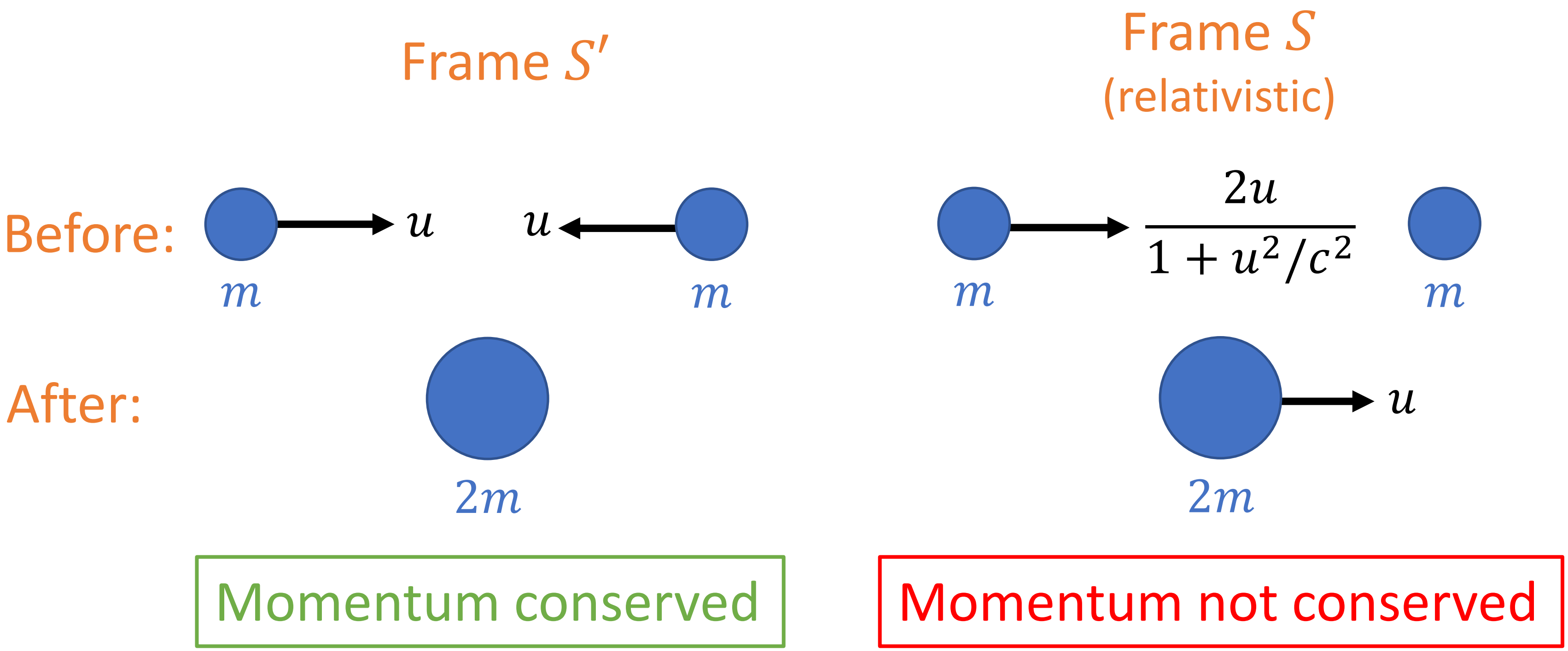


Momentum conserved

Momentum conserved

Modifying Newtonian mechanics

- To preserve conservation laws at high speeds requires us to **modify the definitions of energy and momentum**
- To see why, consider a simple collision in two frames:



Modifying Newtonian mechanics

- This issue is solved by **modifying the definition of the mass of a particle such that it increases with speed u**

- Particle mass $m(u) = \gamma(u)\, m_0 = \frac{m_0}{\sqrt{1-u^2/c^2}}$

- $\gamma(u) = \frac{1}{\sqrt{1-u^2/c^2}}$ is the normal "Lorentz factor" in terms of the speed of the particle, u [not a frame transformation]

- The mass at zero speed, m_0, is called the **rest mass of the particle** and is an invariant in all reference frames

Relativistic momentum

- The **relativistic momentum** of a particle is then defined in the usual way, as mass × velocity [Note in the Workbook]:

$$\text{Relativistic momentum} \quad p(u) = mu = \gamma m_0 u = \frac{m_0 u}{\sqrt{1 - u^2/c^2}}$$

- Does this definition make sense? Show that in the low-u limit ($u/c \ll 1$) we recover the classical definition of momentum, $p = m_0 u$ [maths hint: $(1 + x)^n \approx 1 + nx$]

- The total relativistic momentum is conserved in collisions [we will see some examples soon]

Relativistic energy

- We can use the expression for relativistic momentum to calculate the **kinetic energy gained by a particle** as it accelerates

$$\text{Kinetic energy } T = \int F\,dx = \int \left(\frac{dp}{dt}\right) dx = \int \left(\frac{dx}{dt}\right) dp = \int v \left(\frac{dp}{dv}\right) dv$$

$$\text{Integration by parts:} \quad T = \int_{v=0}^{v=u} v \left(\frac{dp}{dv}\right) dv = [v\,p]_{v=0}^{v=u} - \int_{v=0}^{v=u} p\,dv$$

$$T = \left[\frac{m_0 v^2}{\sqrt{1 - v^2/c^2}}\right]_{v=0}^{v=u} - \left[m_0 c^2 \sqrt{1 - v^2/c^2}\right]_{v=0}^{v=u}$$

$$\text{Kinetic energy } T = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} - m_0 c^2 = [\gamma(u) - 1]\, m_0 c^2$$

Relativistic energy

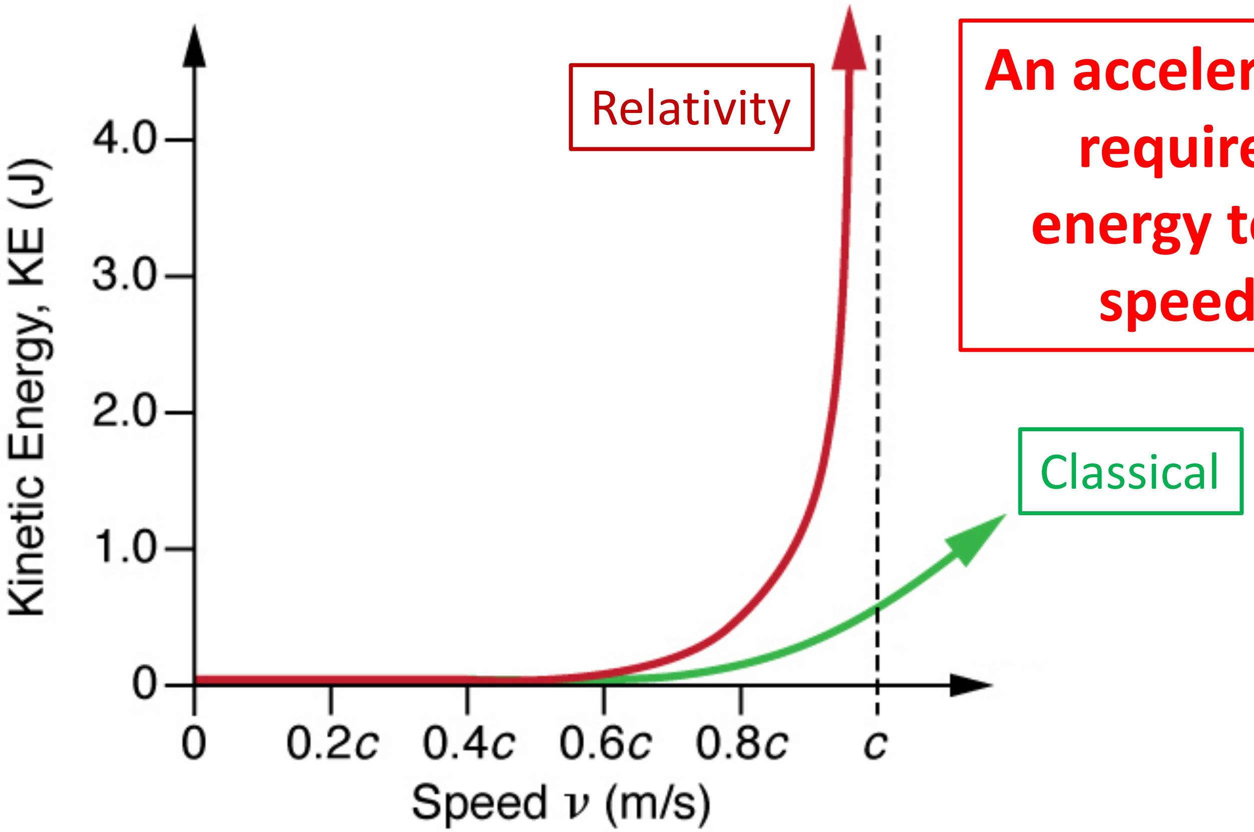
- The **relativistic kinetic energy** of a particle is defined by

$$\text{Kinetic energy} \quad T(u) = [\gamma(u) - 1]\, m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}} - m_0 c^2$$

- Does this definition make sense? Show that in the low-u limit ($u/c \ll 1$) we recover the classical definition of kinetic energy, $T = \frac{1}{2} m_0 u^2$ [maths hint: $(1 + x)^n \approx 1 + nx$]

Relativistic energy

- In relativity, the kinetic energy of a particle $\rightarrow \infty$ as the particle approaches the speed of light, $v \rightarrow c$



An accelerated particle requires infinite energy to reach the speed of light!

https://courses.lumenlearning.com/physics/chapter/28-6-relativistic-energy/

Relativistic energy

- The expression for kinetic energy, $T = [\gamma(u) - 1]\, m_0 c^2$, can be written in terms of a **total energy** E

$$\text{Kinetic energy } T = E(u) - E(0)$$

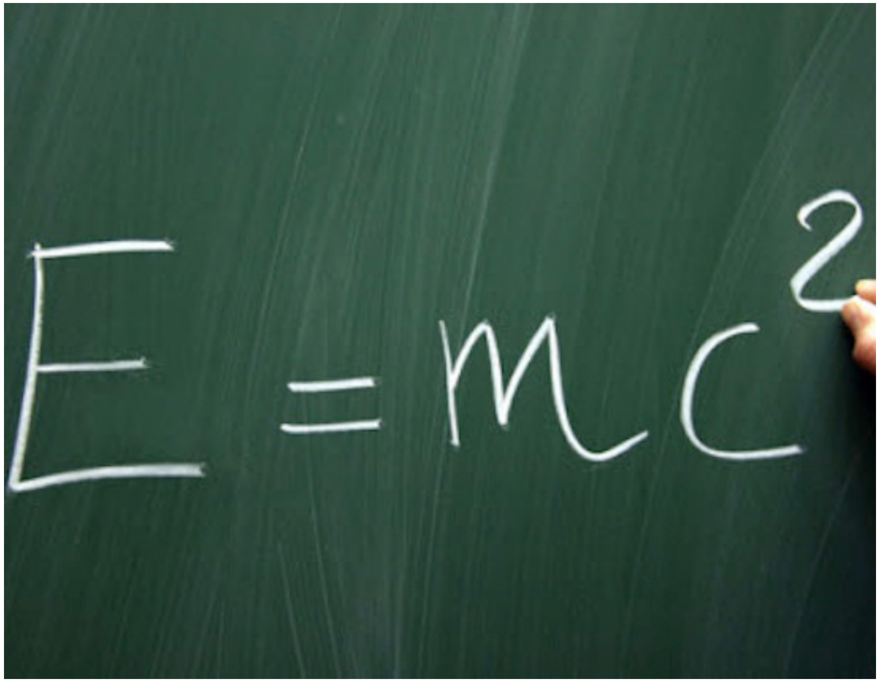
$$\boxed{\text{Total energy } E(u) = \gamma(u)\, m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}}$$

- We conclude that a particle has an **intrinsic rest energy**, $E(0) = m_0 c^2$, such that

$$\boxed{\text{Total energy} = \text{rest energy} + \text{kinetic energy}}$$

Rest-mass energy

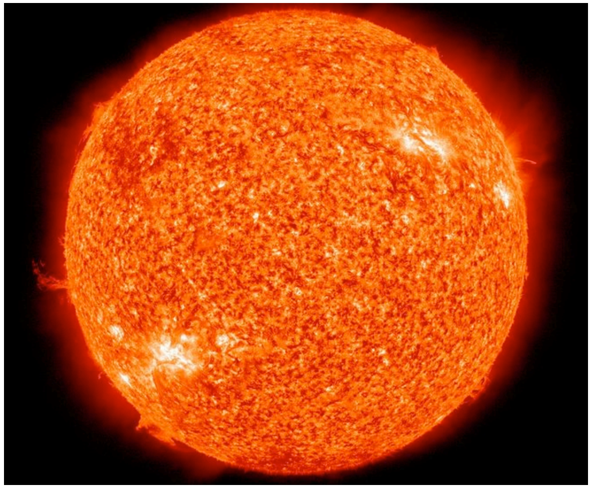
- What is the rest-mass energy of a human?
- Is this a lot of energy, or not?

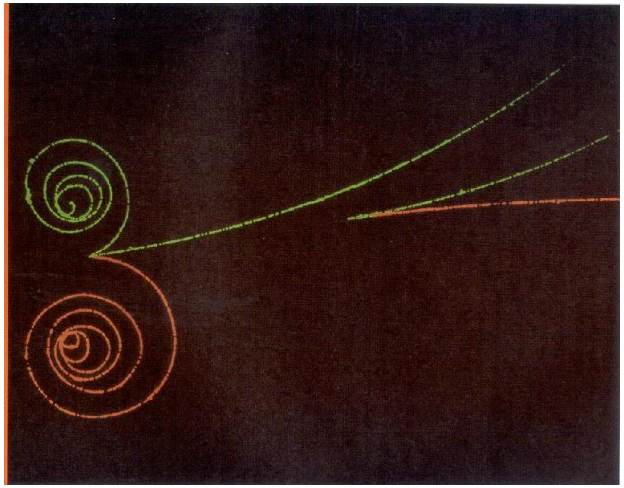


http://explorecuriocity.org/Explore/ArticleId/1606/e-mc-squared-einsteins-relativity-1606.aspx

Rest-mass energy

- The idea that **rest-mass is a form of energy** is one of the most important aspects of relativity, and we will discuss its applications further in the next class





https://www.space.com/19321-sun-formation.html
https://www.elp.com/articles/2015/09/more-nuclear-power-plant-retirements-forecast.html
https://physics.stackexchange.com/questions/56296/why-does-a-photon-colliding-with-an-atomic-nucleus-cause-pair-production

Two useful formulae

- We have seen that in relativity,

$$\text{Momentum} \quad p(u) = \gamma(u) m_0 u = \frac{m_0 u}{\sqrt{1 - u^2/c^2}}$$

$$\text{Energy} \quad E(u) = \gamma(u) m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - u^2/c^2}}$$

- Combining these equations, we find two useful relations:

Energy – rest mass – momentum

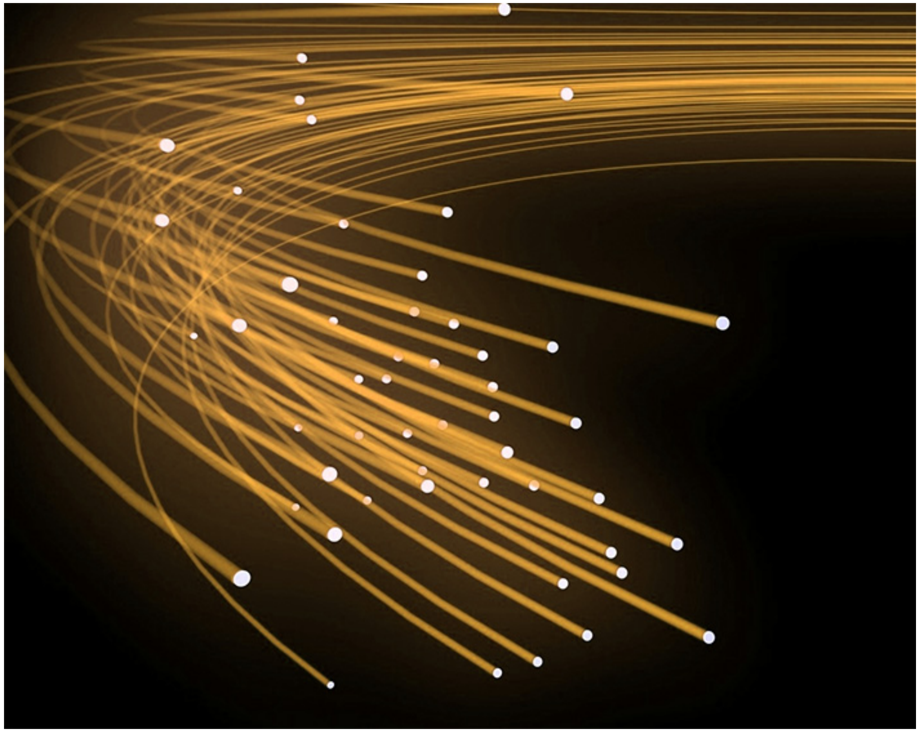
$$E^2 = (m_0 c^2)^2 + (pc)^2$$

Speed – momentum – energy

$$u = \frac{pc^2}{E}$$

Photons

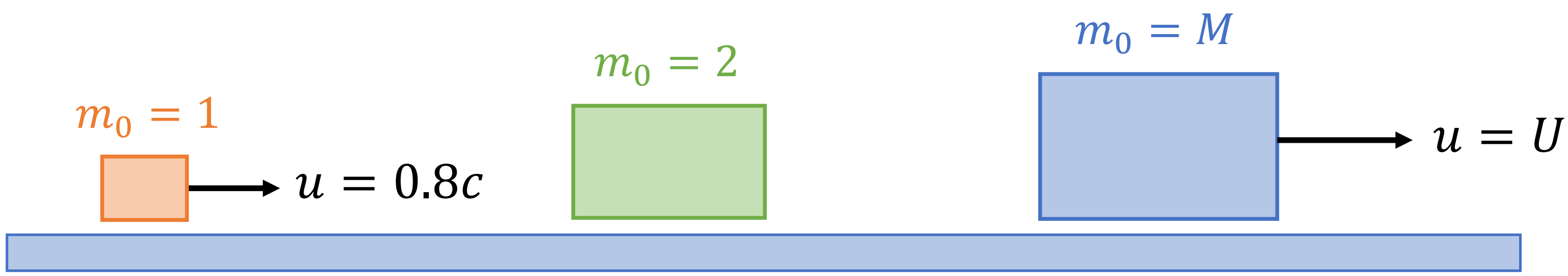
- $E = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$: no particle can travel at the speed of light
- However, quantum mechanics tells us that light itself can behave as particles known as **photons** with energy $E = h\nu$



- This contradiction is resolved if **photons have zero rest-mass,** $m_0 = 0$
- Using $E^2 = (m_0 c^2)^2 + (pc)^2$, we find for photons, $E = pc$
- Momentum $p = \frac{E}{c} = \frac{h\nu}{c}$

Relativistic collision example

- A mass of $1\ kg$, moving at speed $u = 0.8c$, is absorbed by a stationary mass of $2\ kg$. At what speed U does the combined mass recoil? What is the combined mass M?



- Write an equation for the **conservation of momentum**
- Write an equation for the **conservation of energy**
- Eliminate variables between these two equations to solve for the two unknowns U and M

Relativistic collision example

- Checking your answers: $U = \frac{4}{11}c = 0.36c, M = 3.42\ kg$
- As this example shows, **mass is not conserved in collisions**!! (rest-mass before $= 1 + 2 = 3$, rest-mass after $= 3.42$). Rather, **energy and momentum are conserved**
- *Question: where does the extra mass come from?*

Class 5: Energy and Momentum

At the end of this session you should be able to …

- … recall how the definitions of **momentum** and **energy** are modified in relativity
- … apply **conservation laws** of these quantities, to study relativistic particle collisions
- … be familiar with the concept of the **rest mass** of a particle, and how it represents an **equivalent energy**
- ... understand how to calculate the energy and momentum of **photons**, particles travelling at the speed of light