Class 4: Accelerated Motion

In this class we will investigate how accelerated motion can be studied within Special Relativity; in the next class we'll discuss how this is equivalent to motion in a gravitational field

Class 4: Accelerated Motion

At the end of this session you should be able to …

- … determine the **world line of an accelerated object** in special relativity
- … represent this motion on a space-time diagram
- … describe how the **clock rate varies** within a long rocket ship undergoing accelerated motion

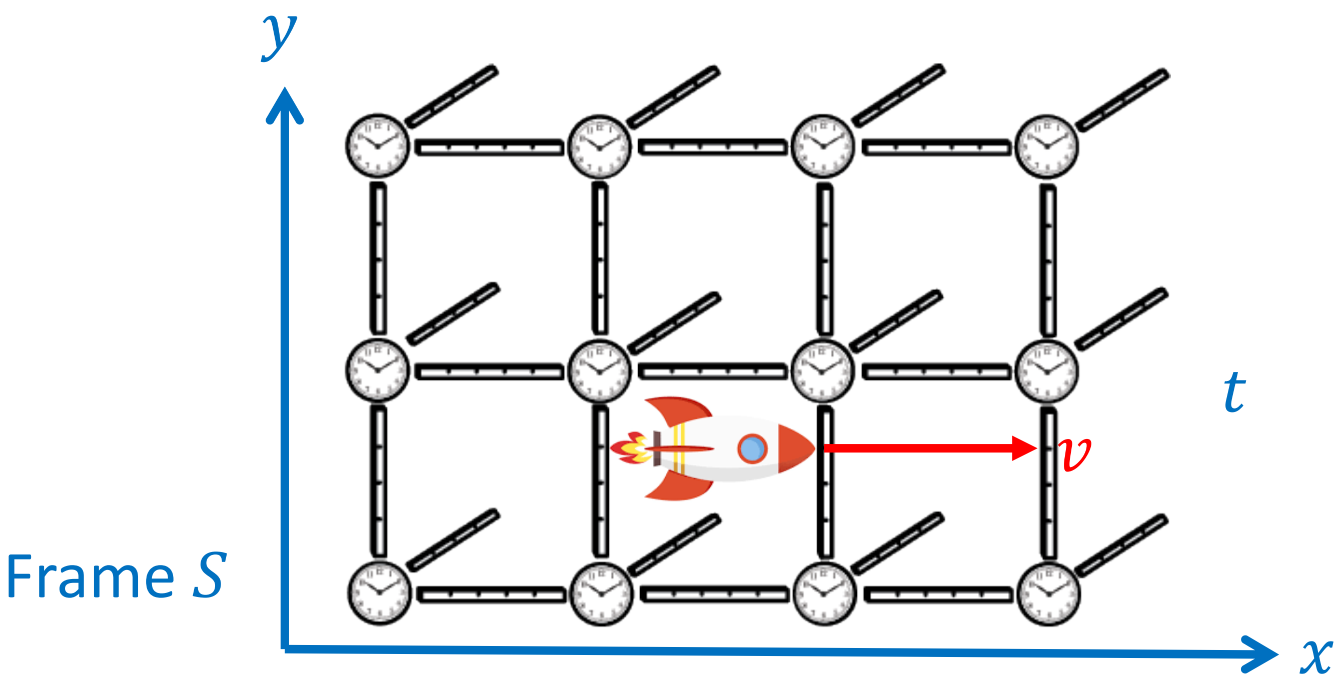
Analyzing acceleration in SR

- One of the centre-pieces of classical physics is Newton's 2nd Law: $\vec{F} = m\vec{a}$

- What does acceleration look like in Special Relativity?

Analyzing acceleration in SR

- What does uniform acceleration look like in frame S?



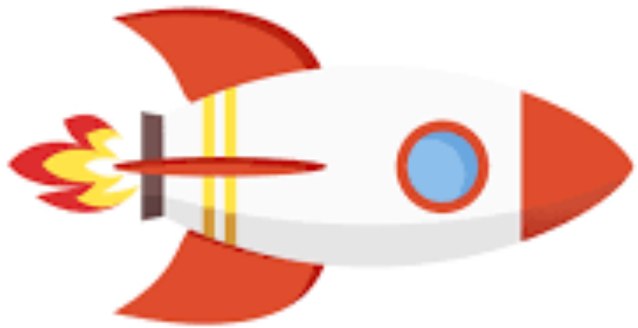
- Initial guess : $\frac{dv}{dt} = \text{constant} \rightarrow v = v_0 + at$
- Does this make sense??

Analyzing acceleration in SR

- No – velocity will eventually exceed c!
- No – "dv" involves subtracting velocities at nearby points, but we know velocities add in strange ways in relativity!
- No – "dt" involves time intervals, but these vary between frames
- $\frac{dv}{dt} =$ constant is unphysical and can't hold in all frames

Analyzing acceleration in SR

- Let’s define uniform acceleration in a different way – as **“feeling constant to the object being accelerated”**



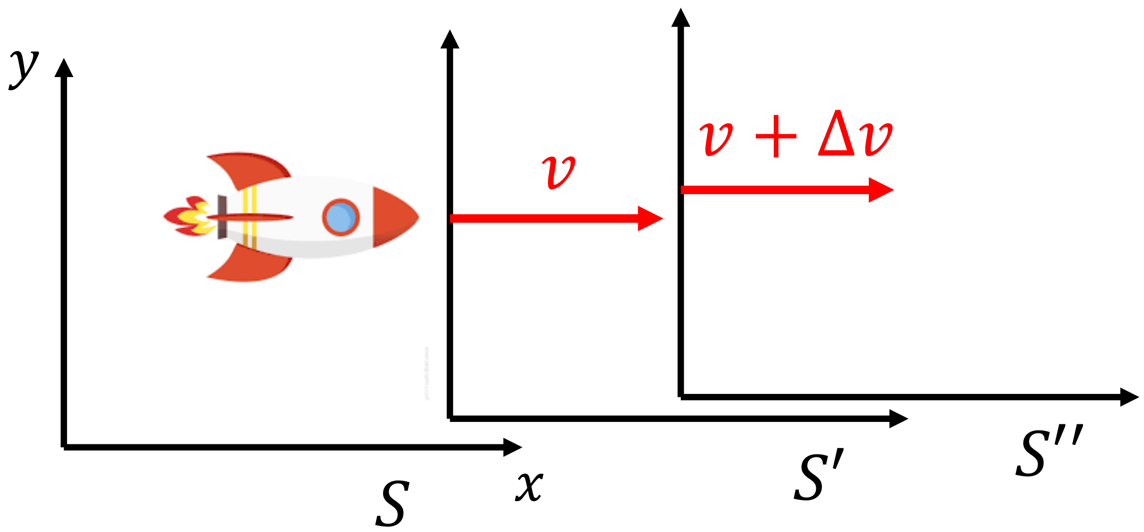
https://www.decodedscience.org/jet-airplanes-take-off-speeds-and-aircraft-performance/5453

- (The accelerated observer can always measure this rate by gently dropping a rock out of the window)

Analyzing acceleration in SR

- How can we analyze this in terms of inertial frames?

- *Set up a large number of inertial frames with a range of relative velocities, such that the accelerating object is temporarily at rest in each – **analyze the motion by patching all the frames together***



Analyzing acceleration in SR

- We define **constant proper acceleration** by

$$\alpha = \frac{dv'}{d\tau} = \text{constant}$$

$d\tau =$ proper time elapsed on the rocket-ship clock in a small interval ($= dt'$ in S')

$dv' =$ momentary increase in speed from rest in S' (non-relativistic, since initially $v' = 0$)

World line of accelerating object

- We can determine the space-time co-ordinates $x(t)$ of the rocket-ship in the original frame S, in terms of the proper time co-ordinate τ in the ship's rest frame

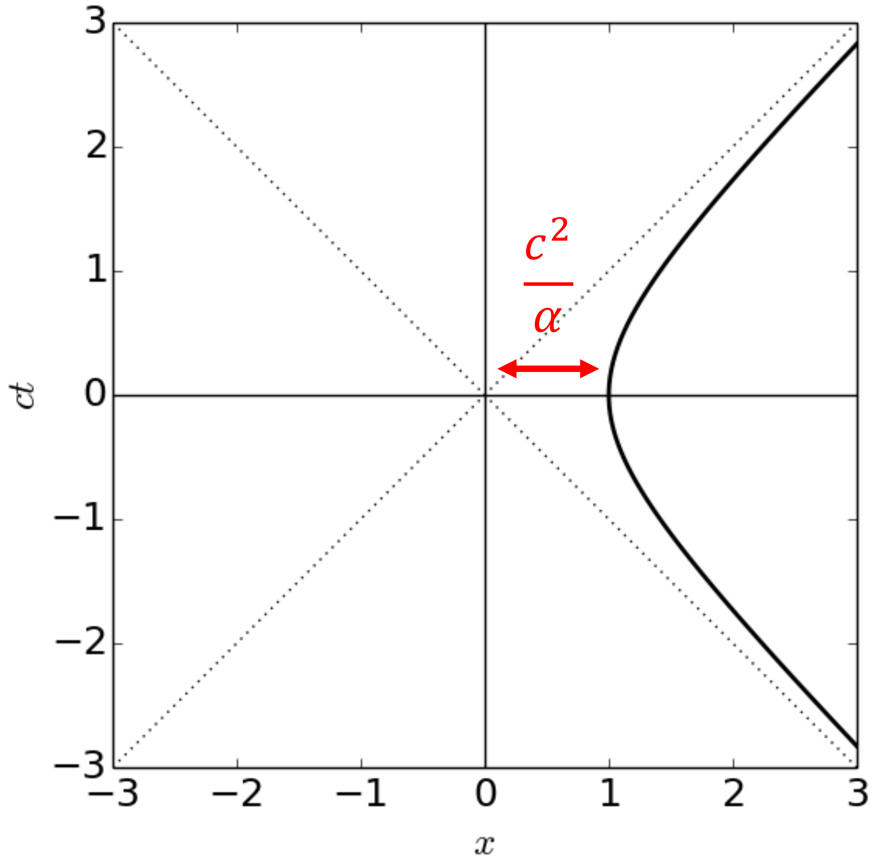
- The velocity in S is $\boldsymbol{v = c \tanh\left(\frac{\alpha\tau}{c}\right)}$ – at large times $v \to c$, but the ship's speed never exceeds c

- The t co-ordinate in S is $\boldsymbol{t = \frac{c}{\alpha} \sinh\left(\frac{\alpha\tau}{c}\right)}$

- The x co-ordinate in S is $\boldsymbol{x = \frac{c^2}{\alpha} \cosh\left(\frac{\alpha\tau}{c}\right)}$

Accelerating space-time diagram

- Uniformly-accelerated objects are moving along a hyperbola in a space-time diagram, $x^2 - (ct)^2 = (c^2/\alpha)^2$

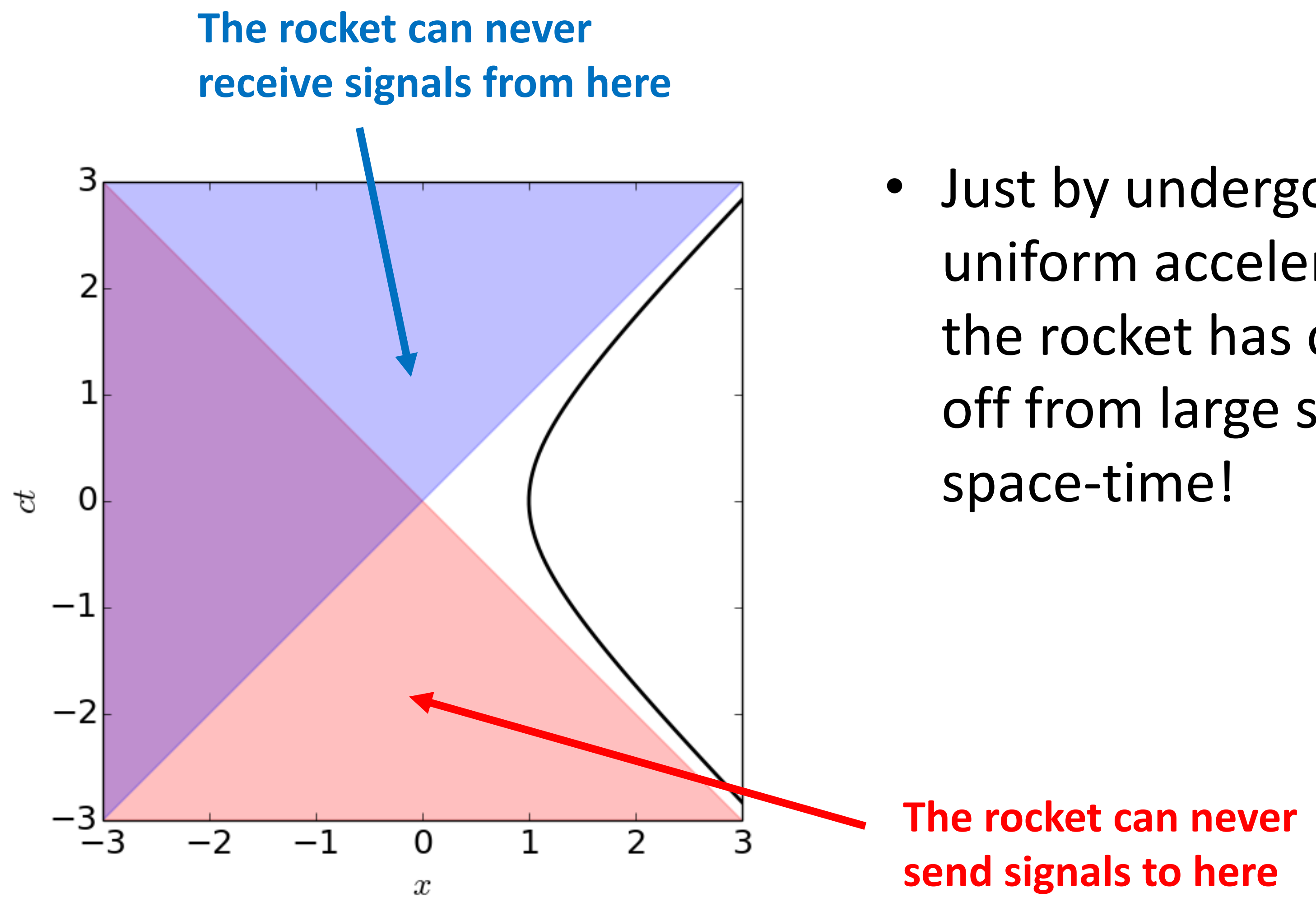


- The object remains a **constant proper distance** from the origin (sounds confusing, but this is a distance in space-time, not space!)

- The object's space-time diagram is **the same in every frame** (whenever it enters my frame at $v = 0$, I find its acceleration to be α)

Accelerating space-time diagram

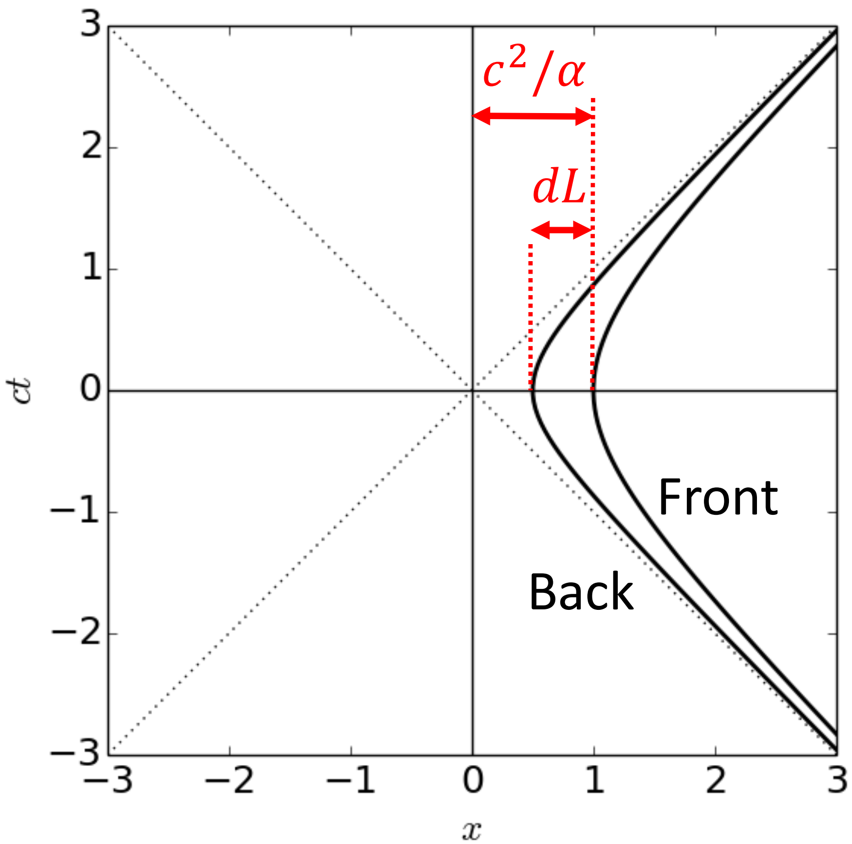
- Consider this rocket **sending and receiving light signals**



- Just by undergoing uniform acceleration, the rocket has cut itself off from large sections of space-time!

Clock rates in a rocket ship

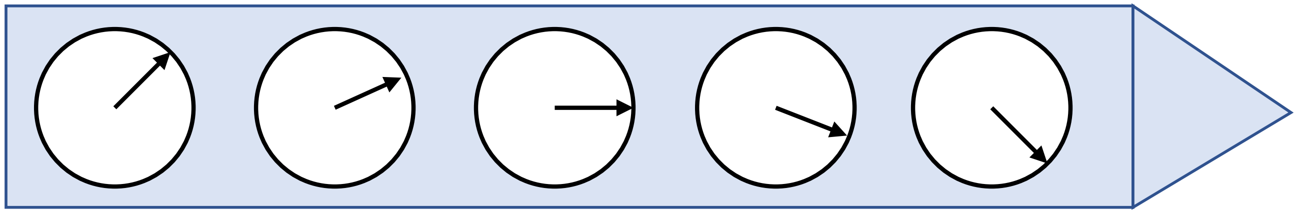
- Now consider a "long rocket", such that its front and back have different world lines, but are separated by a fixed proper length dL at any instant (i.e., $dx = dL$ when $dt = 0$)



- The proper accelerations of the front and back must be different!
- $\frac{c^2}{\alpha_F} - \frac{c^2}{\alpha_B} = dL$
- $\alpha_F < \alpha_B$

Clock rates in a rocket ship

- Synchronize the front and back observer clocks at $\tau = 0$, when the rocket is instantaneously at rest in S
- At some later instant when the rocket is at rest in S', $v = c\tanh(\alpha\tau/c)$ for both ends. But α's are different! $\rightarrow \frac{\Delta\tau_F}{\Delta\tau_B} = \frac{\alpha_B}{\alpha_F}$
- More proper time has elapsed at the front of the rocket! **The front clocks are running "faster"!**



- This phenomenon occurs in *the accelerating frame of the long rocket* – it's **not an inertial frame**, so does not contradict SR