Class 3: Lorentz Transformations

In this class we will explore the Lorentz transformations, the equations which relate the space-time co-ordinates of an event viewed in different reference frames

Class 3: Lorentz Transformations

At the end of this session you should be able to …

- … be familiar with the **Lorentz transformations**, which relate the space-time co-ordinates of an event in different frames
- … apply the Lorentz transformations to suitable events to derive the phenomena of **time dilation** and **length contraction**
- … understand the significance of the **space-time interval** ds^2 between two events and the idea of **invariants**
- … apply the law of **combination of velocities** in relativity

Lorentz transformations

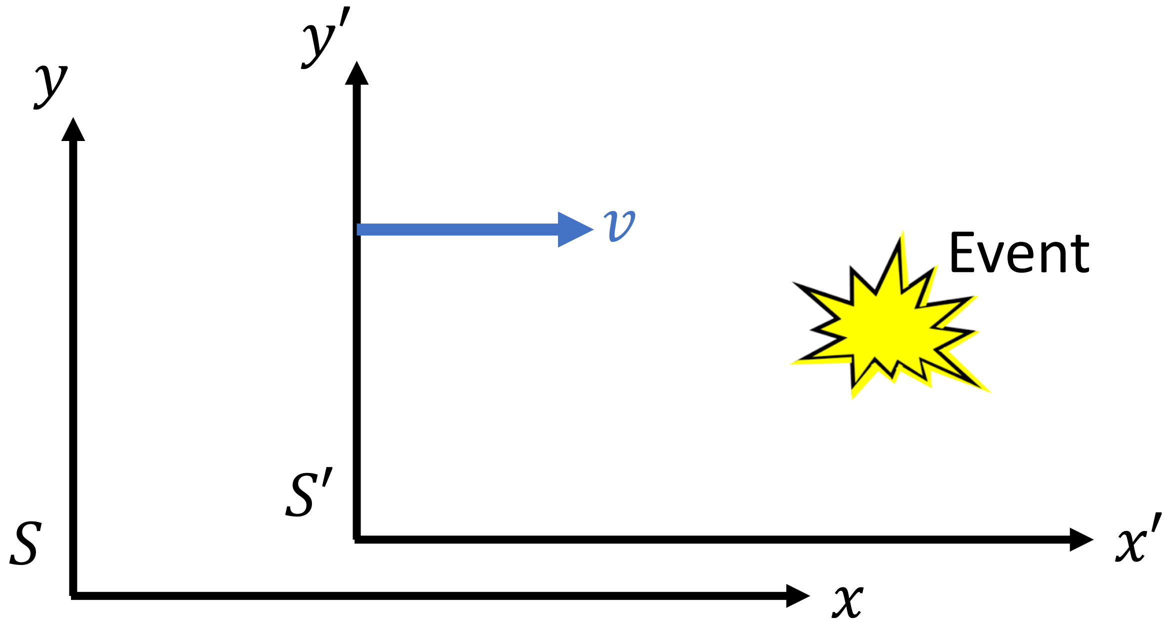
- In the previous class, we encountered some of the remarkable phenomena of relativity, such as **time dilation** and **length contraction**
- We will now learn how to derive these effects more systematically, using the Lorentz transformations

H.Lorentz (1853-1928)

https://www.nobelprize.org/nobel_prizes/physics/laureates/1902/lorentz-bio.html

Lorentz transformations

- An **event** is a physical occurrence at a unique point in space and time



- The Lorentz transformations provide the **algebraic relations for relating the co-ordinates of this event** in two different inertial reference frames, S and S'

Lorentz transformations

Lorentz transformations from S to S'

$$t' = \gamma(t - vx/c^2)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

Inverse transformations from S' to S

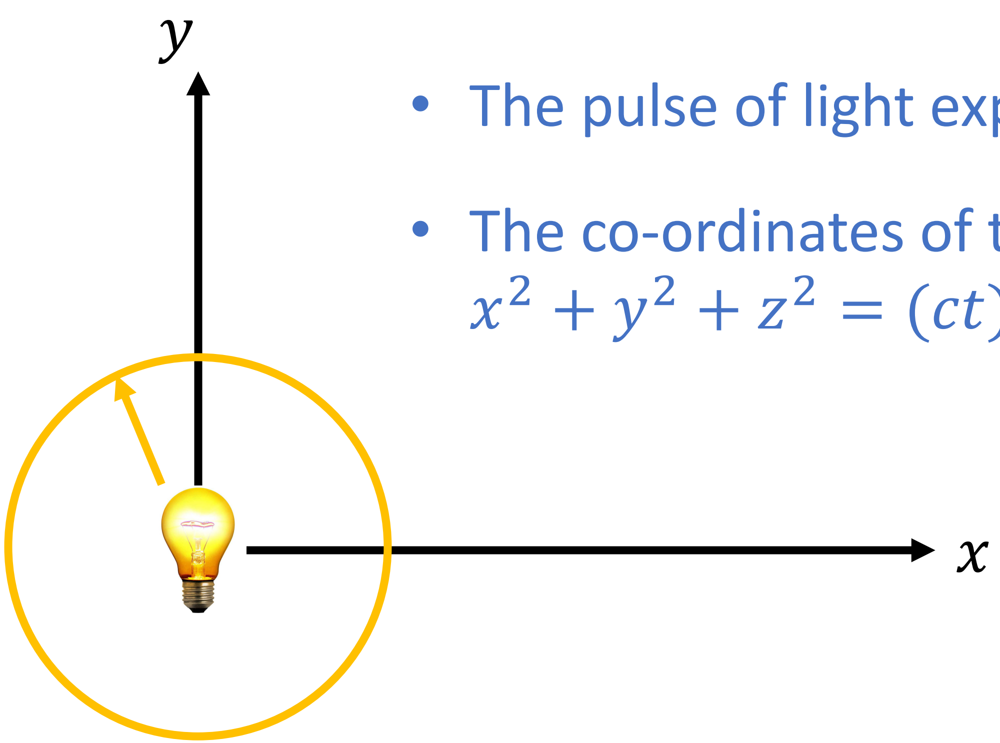
$$t = \gamma(t' + vx'/c^2)$$
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

[Note this in the Workbook]

Lorentz transformations

- These transformations work because they **preserve the constancy of the speed of light** in different frames
- Consider a flash of light emitted from the origin of S



- The pulse of light expands in a **sphere**
- The co-ordinates of the wavefront are $x^2 + y^2 + z^2 = (ct)^2$

Lorentz transformations

- What are the co-ordinates of the wavefront in S'?
- Substitute in the Lorentz transformations:

$$x^2 + y^2 + z^2 = (ct)^2$$

$$\gamma^2(x' + vt')^2 + y'^2 + z'^2 = c^2\gamma^2(t' + vx'/c^2)^2$$

$$\gamma^2 x'^2 + \cancel{2\gamma^2 x' v t'} + \gamma^2 v^2 t'^2 + y'^2 + z'^2$$
$$= \gamma^2 c^2 t'^2 + \cancel{2\gamma^2 x' v t'} + \gamma^2 v^2 x'^2/c^2$$

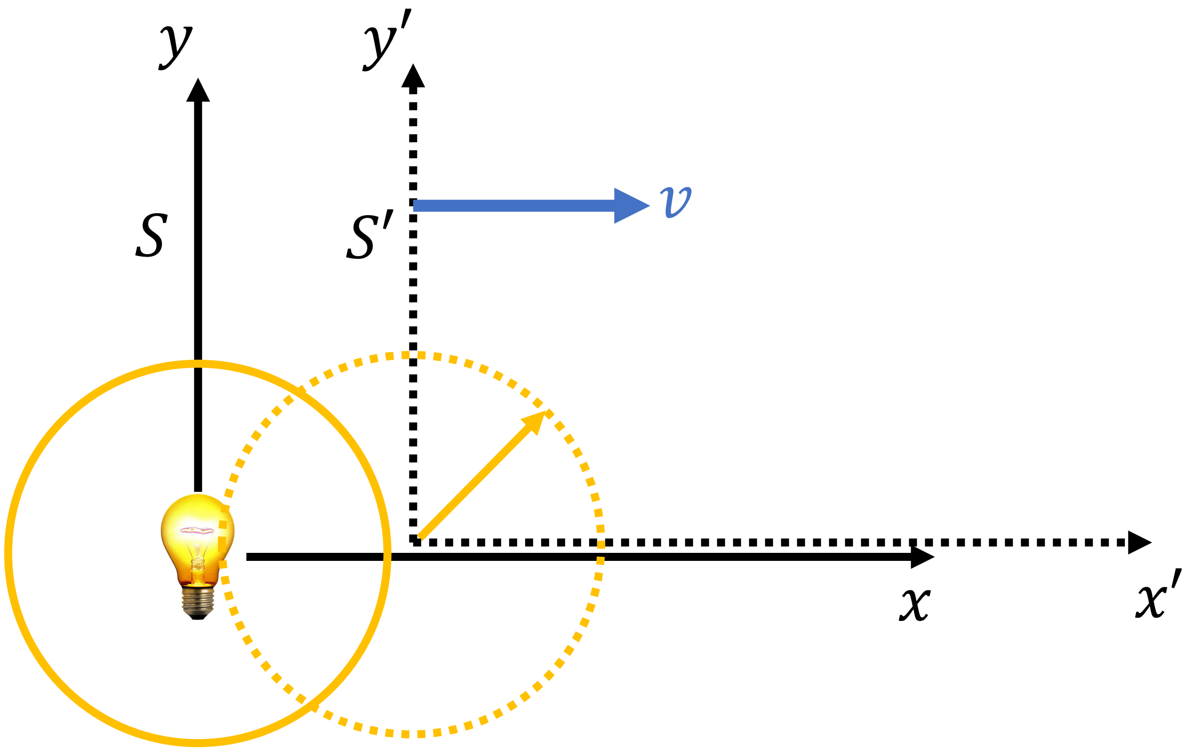
$$\gamma^2 x'^2 (1 - v^2/c^2) + y'^2 + z'^2 = \gamma^2 c^2 t'^2 (1 - v^2/c^2)$$

$= 1/\gamma^2$ $\qquad$ $= 1/\gamma^2$

$$\boldsymbol{x'^2 + y'^2 + z'^2 = (ct')^2}$$

Lorentz transformations

- The wavefront also expands as a sphere, with speed c, in S'



- The Lorentz transformations are consistent with Einstein's postulate of special relativity

Lorentz transformations

- In the Lorentz transformations – unlike in the Galilean transformations – there is **no such thing as absolute time**

Lorentz transformation (correct)

$$\boldsymbol{t' = \gamma(t - vx/c^2)}$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

Galilean transformation (incorrect)

$$\boldsymbol{t' = t}$$

$$x' = x - vt$$

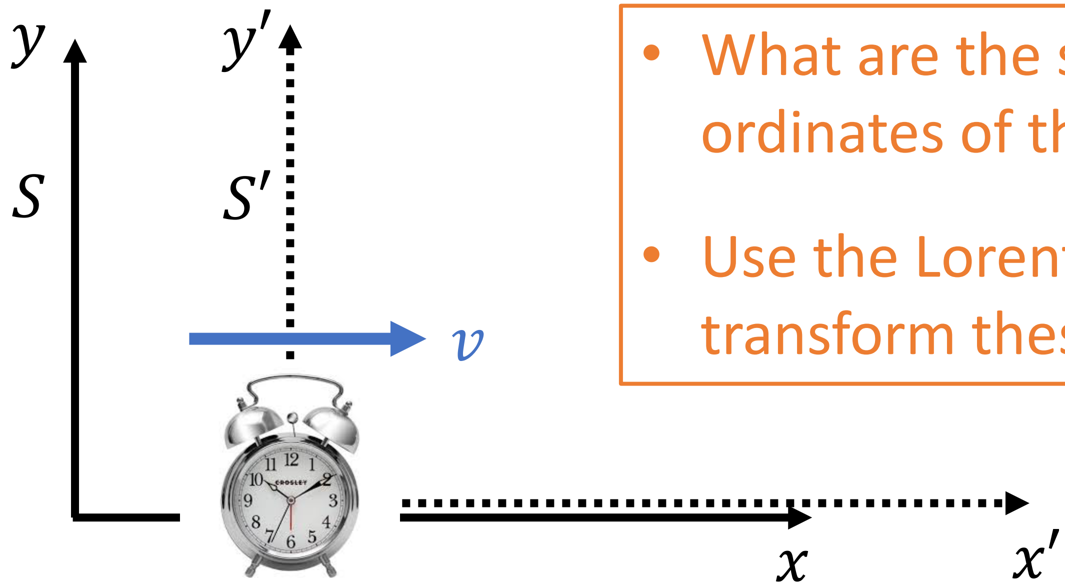
$$y' = y$$

$$z' = z$$

Time dilation re-visited

- We apply the Lorentz transformations by **identifying events and their co-ordinates**. Let's try some examples.

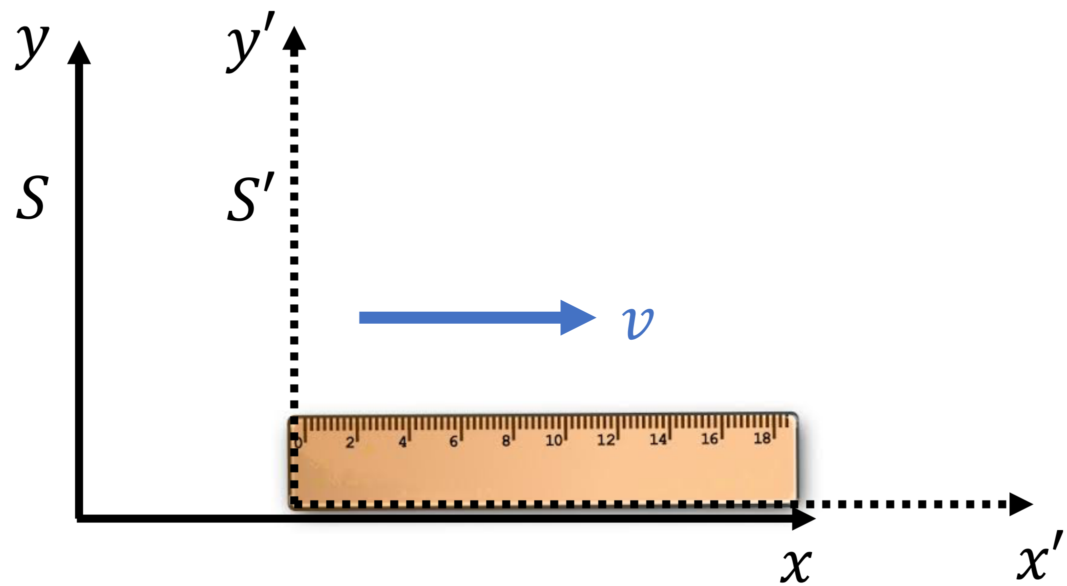
- Consider a clock at rest at the origin of S', which ticks at times $t' = 0$ and $t' = \tau$. What is the time difference between these ticks as measured in S?



- What are the spacetime co-ordinates of the two ticks in S'?
- Use the Lorentz transformations to transform these co-ordinates to S

Length contraction re-visited

- Consider a ruler at rest in S', with ends at $x' = 0$ and $x' = L$



- Observers in S measure the length of the ruler by **marking the positions of the two ends simultaneously in S**, at time t. What is the length of the ruler as measured in S?

Space-time interval

- Consider two events, which are separated in space-time by co-ordinate differences $(\Delta x, \Delta y, \Delta z, \Delta t)$ in S
- A special quantity is the **space-time interval** between the events, which has the symbol Δs^2

Space-time interval

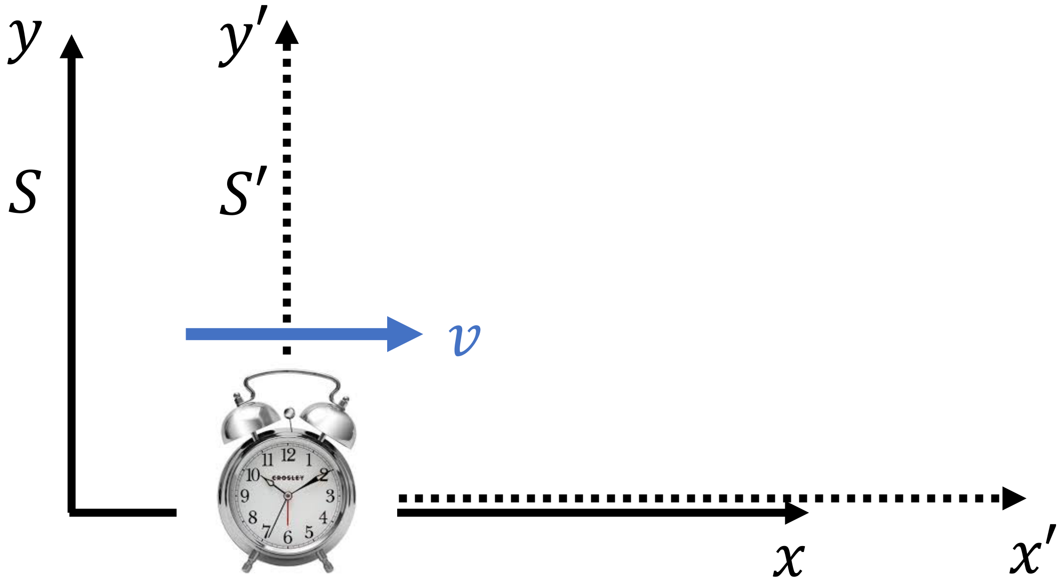
$$\boldsymbol{\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2}$$

- If observers in S' compute the space-time interval of the same events using their co-ordinate differences, $(\Delta x', \Delta y', \Delta z', \Delta t')$, they will find **exactly the same answer**! [Note in the Workbook]

$$-c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -c^2 \Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$$

Space-time interval

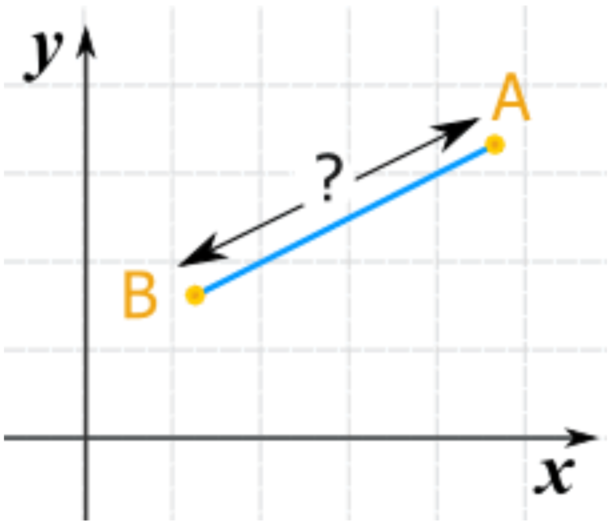
- Let's prove that is the case by considering two ticks of a clock at rest at the origin of S', whose events we computed earlier



- What is the space-time interval between two ticks of the clock as determined by S and S'?

Space-time interval

- A quantity such as Δs^2, which has the same value in all reference frames, is called an **invariant**
- *Quantities which all observers agree to be the same are useful!*
- In Euclidean geometry, the distance between 2 points is the same if you rotate co-ordinates



https://www.mathsisfun.com/algebra/distance-2-points.html

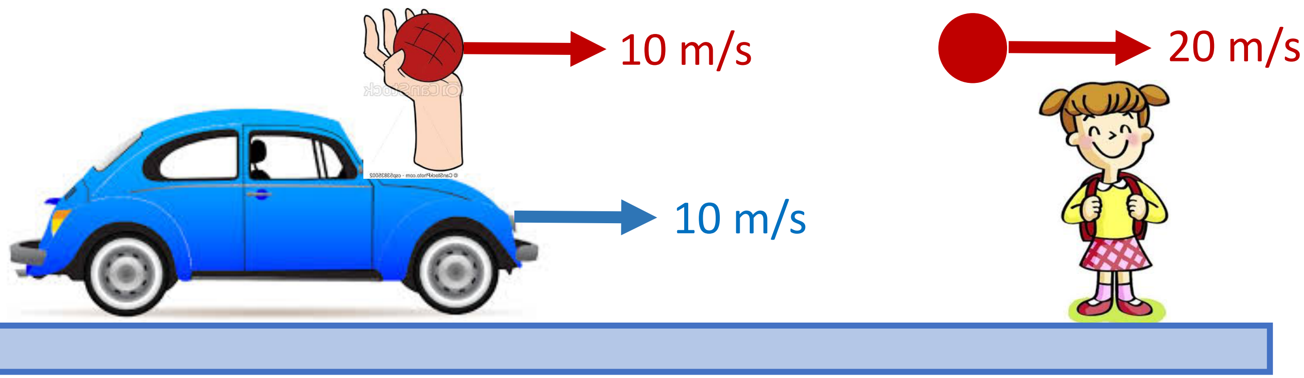
The **space-time interval in relativity** (same in all inertial frames) is analogous to a **distance in Euclidean geometry** (same in all rotated frames)

Space-time interval

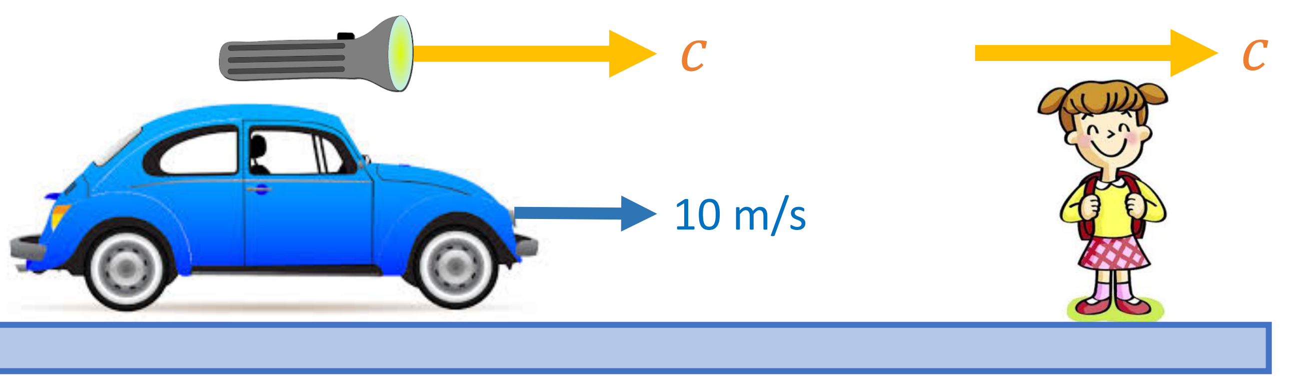
- The space-time interval, $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2$, between 2 events can be **zero**, **negative** or **positive** [ignoring y, z for clarity]
- If $\Delta s^2 = 0$, then the 2 events can be **linked by a light signal** ($\Delta x = c\,\Delta t$) in any frame
- If $\Delta s^2 < 0$, then an **observer can pass through them both** (there is a frame with $\Delta x = 0$) but there is no frame in which the events occur simultaneously ($\Delta t = 0$)
- If $\Delta s^2 > 0$, there is **a frame in which the events occur simultaneously**, but none in which they occur at the same place
- *(We'll discuss these points further in the next class)*

Combination of velocities

- In classical physics, relative velocities add and subtract



- In relativity, Einstein's postulate shows this is wrong



Combination of velocities

- We can use the Lorentz transformations to derive the exact formula. First convert to the differentials:

$$x = \gamma(x' + vt') \quad \rightarrow \quad dx = \gamma(dx' + v\,dt')$$

$$t = \gamma(t' + vx'/c^2) \quad \rightarrow \quad dt = \gamma(dt' + v\,dx'/c^2)$$

- Now divide these two relations to find the speed $u_x = \frac{dx}{dt}$ in S, of an object moving with speed $u'_x = \frac{dx'}{dt'}$ in S':

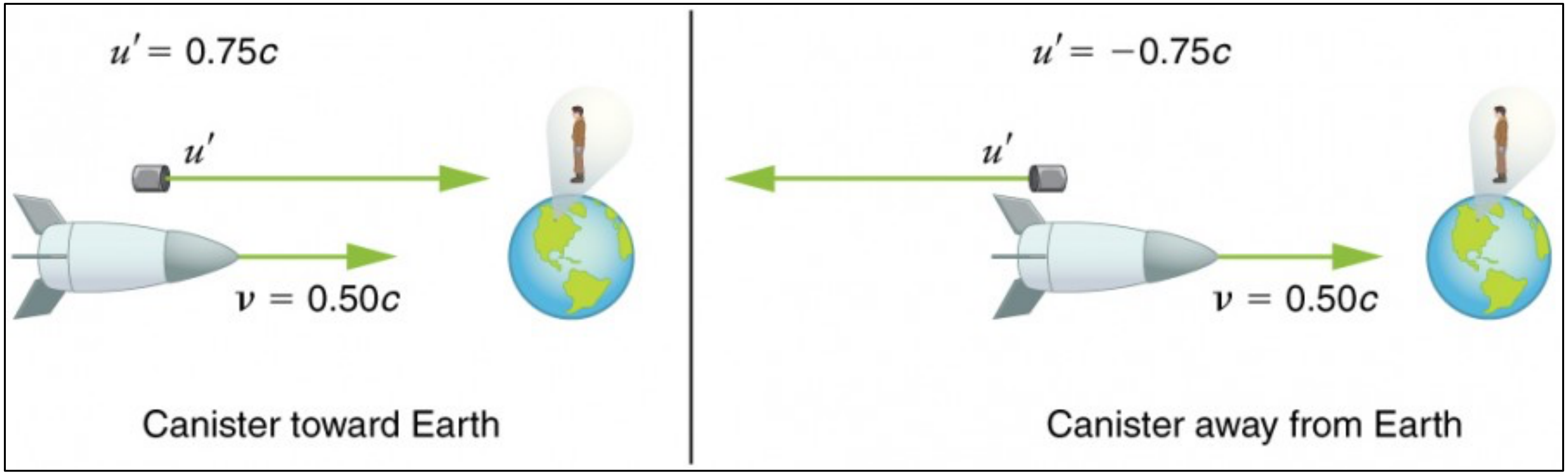
$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

- Does this work? Yes – **when $\boldsymbol{u'_x = c}$, then $\boldsymbol{u_x = c}$**

Combination of velocities

Combination of velocities: $u = \dfrac{u' + v}{1 + u'v/c^2}$ $u' = \dfrac{u - v}{1 - uv/c^2}$

- The Amazon-space delivery service, a rocket moving at speed $v = 0.5c$, fires a package towards the Earth at speed $u' = 0.75c$ in the rocket frame. What is the package speed as measured by the Earth observer?



- How does your answer change if the package is accidentally fired in the opposite direction, at the same speed in the rocket frame?

[Example adapted from https://courses.lumenlearning.com/physics/chapter/28-4-relativistic-addition-of-velocities/]

Combination of velocities

- **How about velocities in the perpendicular direction?**
- Although $y = y'$, $u_y \neq u'_y$!! Carrying out a similar derivation to before …

$$y = y' \quad \rightarrow \quad dy = dy'$$

$$t = \gamma(t' + vx'/c^2) \quad \rightarrow \quad dt = \gamma(dt' + v\,dx'/c^2)$$

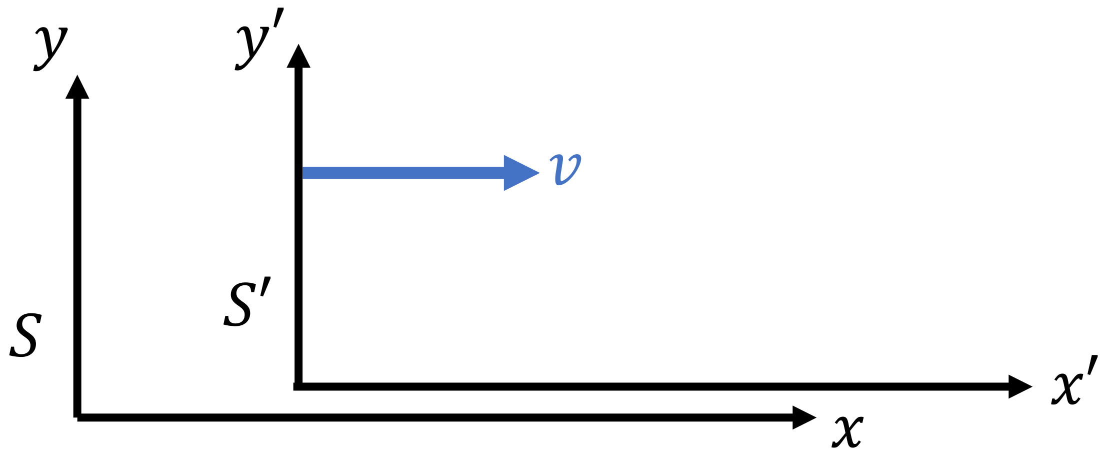
- Dividing these two relations,

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)}$$

- The y-velocity is altered due to **time dilation**

The same, or not the same?

Which of the following quantities are **necessarily the same** when measured in two inertial frames, S and S'? Which are not?



- Value of the speed of light in a vacuum
- Speed of an electron
- Value of the charge on an electron
- Kinetic energy of a proton
- Value of the electric field at a point
- Time between two events
- Order of elements in the periodic table
- Newton's First Law of Motion

[Example from Taylor & Wheeler, Spacetime Physics, Sample Problem 3.1]

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