Class 3: Electromagnetism

In this class we will apply index notation to the familiar field of electromagnetism, and discuss its deep connection with relativity

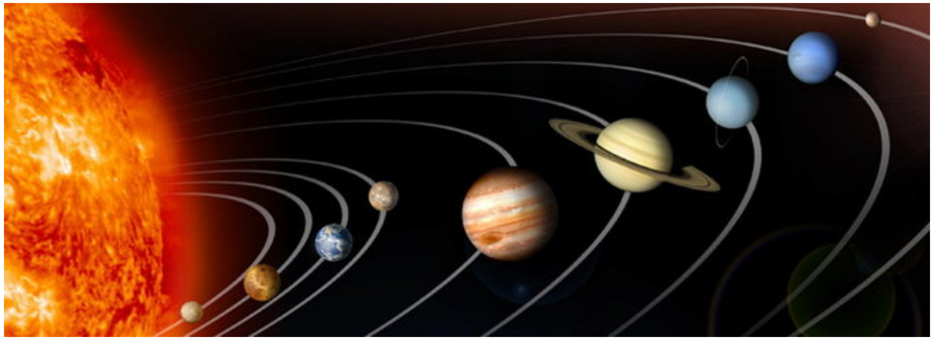
Class 3: Electromagnetism

At the end of this session you should be able to …

- … describe how electromagnetism is **fundamentally a relativistic phenomenon**
- … understand how electromagnetism can be formulated in index notation using the **Maxwell Field Tensor**
- … use the Lorentz transformations to **transform electric and magnetic fields between frames**
- … describe **electromagnetic energy density and flow** in terms of index notation

Synthesis of ideas

- Physics is about the **synthesis of ideas**: understanding apparently different phenomena as the joint consequences of a deeper reality



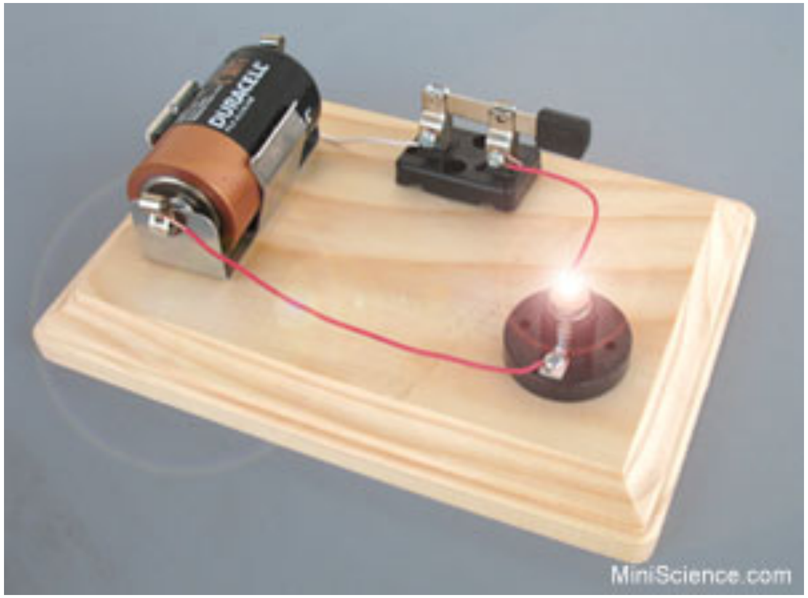
https://www.pinterest.co.uk/pin/477451997975217238
https://www.space.com/16080-solar-system-planets.html

For example, in 1666 Newton realized that the same gravitational force causes both apples to fall downwards on the Earth, and the Earth to orbit the Sun

Synthesis of ideas

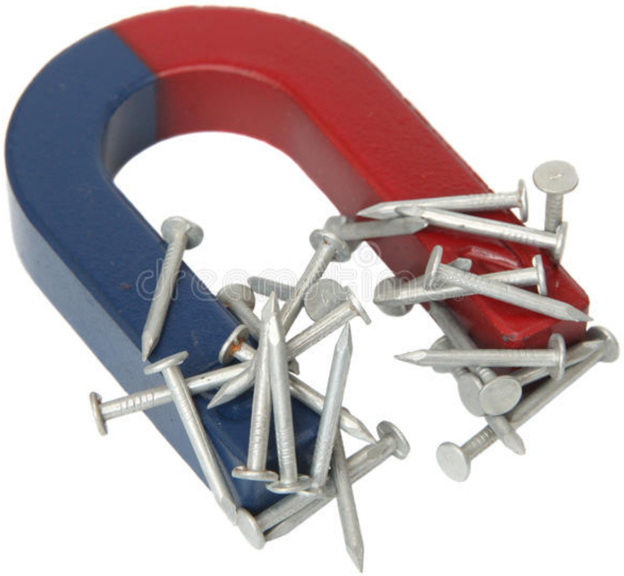
- **Electromagnetism** represents another great synthesis of ideas. Prior to 1830, *electricity* and *magnetism* were considered separate phenomena

Electricity



http://www.miniscience.com/kits/KITSEC/index.html

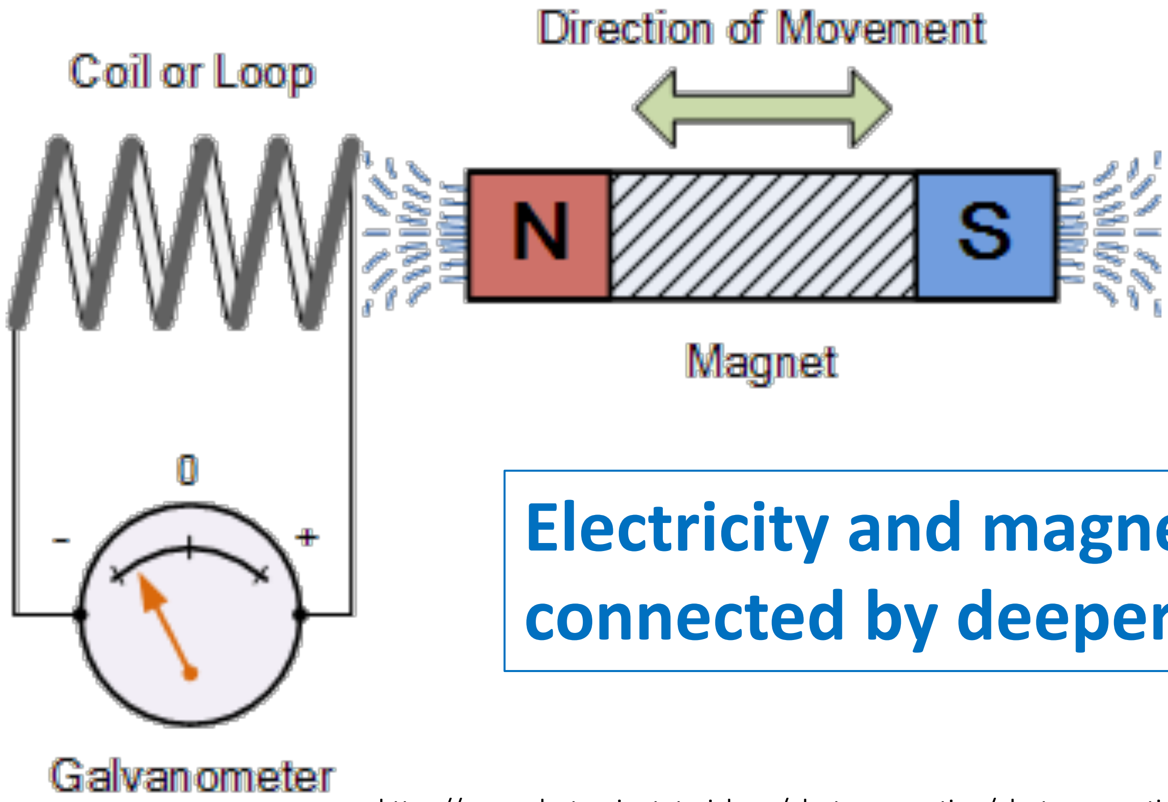
Magnetism



https://www.dreamstime.com/stock-image-magnet-nails-image29533081

Synthesis of ideas

- However, Faraday's experiments demonstrated that *a magnet, as well as a battery, can drive a current*

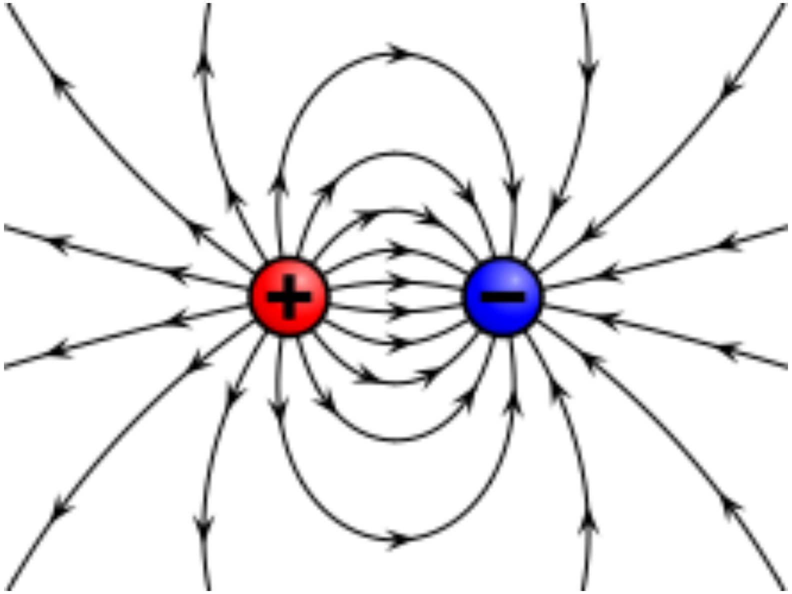


Electricity and magnetism are connected by deeper principles!

https://www.electronics-tutorials.ws/electromagnetism/electromagnetic-induction.html

Electric field

- Electric charge generates an **electric field** $\vec{E}$ in the surrounding region

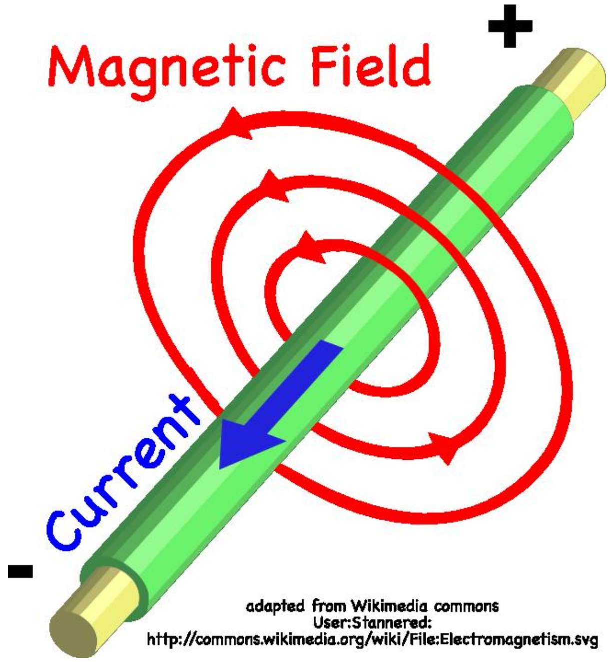


https://en.wikipedia.org/wiki/Electric_field

- *An electric field is a region of space in which an electric charge feels a force,* $\vec{F} = q\vec{E}$

Magnetic field

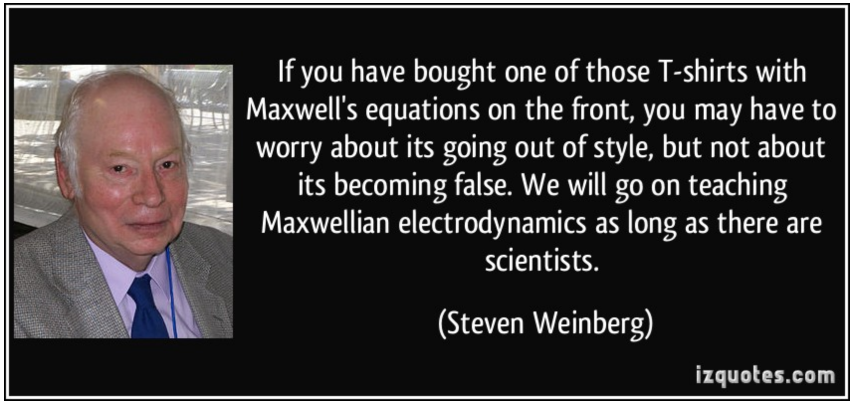
- Electric current (or moving charge) generates a **magnetic field** $\vec{B}$ in the surrounding region



- *A magnetic field is a region of space in which an electric current feels a force,* $\vec{F} = I\, d\vec{l} \times \vec{B}$

Maxwell’s Equations

- **Maxwell’s Equations** describe how $\vec{E}$ and $\vec{B}$-fields are generated from charge and current



http://www.hotelsrate.org/maxwell-equation-t-shirt/
http://izquotes.com/quote/277247

Electromagnetic potentials

- **Potentials**: $\vec{E}$ and $\vec{B}$-fields can be generated from an *electrostatic potential* V and *magnetic potential* $\vec{A}$

- Electrostatics: $\vec{\nabla}\times\vec{E} = \vec{0} \rightarrow \boldsymbol{\vec{E} = -\vec{\nabla}V}$

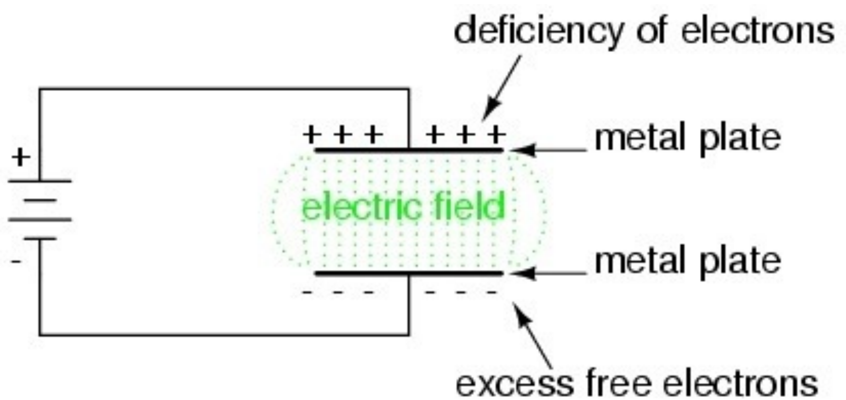
- Magnetism: $\vec{\nabla}.\vec{B} = 0 \rightarrow \boldsymbol{\vec{B} = \vec{\nabla}\times\vec{A}}$

https://www.pinterest.com.au/pin/512143788857285159/

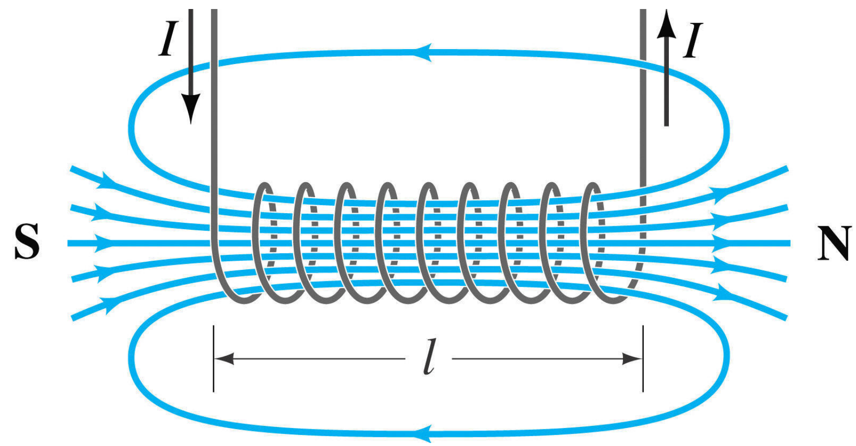
- Time-varying: $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0} \rightarrow \boldsymbol{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}}$

Electromagnetic energy

- **Energy**: it takes work to establish an $\vec{E}$ or $\vec{B}$-field



https://www.quora.com/Why-doesnt-a-dielectric-change-the-electric-field-generated-by-a-capacitor-V-constant



https://www.miniphysics.com/ss-magnetic-field-due-to-current-in-a-solenoid.html

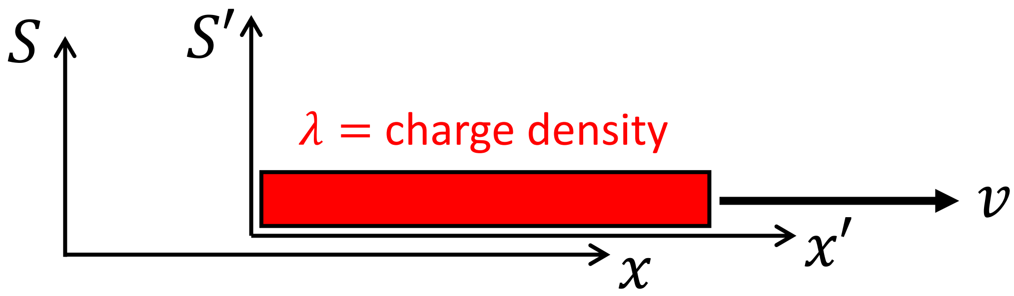
- We may think of this work as being **stored in the fields** with energy density $\frac{1}{2}\varepsilon_0 E^2$ or $\frac{1}{2\mu_0}B^2$

- The **flux of energy** is given by $\frac{1}{\mu_0}\frac{\vec{E}\times\vec{B}}{c}$ (the "Poynting vector")

4-vectors for electromagnetism

- Since $\vec{E}$ is produced by charges, and $\vec{B}$ by moving charges, it's natural that $\vec{E}$ and $\vec{B}$ are **linked by viewing in different reference frames** – i.e., by a Lorentz transformation

- When a line of charge moves, it becomes a current!



4-vectors for electromagnetism

- Since $\vec{E}$ is produced by charges, and $\vec{B}$ by moving charges, it's natural that $\vec{E}$ and $\vec{B}$ are **linked by viewing in different reference frames** – i.e., by a Lorentz transformation
- However, to develop electromagnetism in relativistic notation, we need some 4-vectors!
- In the last class we saw that **the sources of the fields form a 4-vector**, $J^\mu = (\rho c, J_x, J_y, J_z)$ with continuity $\partial_\mu J^\mu = 0$
- In this class we will demonstrate that electromagnetism may be described very nicely if **the potentials also form a 4-vector**, $A^\mu = (\frac{V}{c}, A_x, A_y, A_z)$

Maxwell field tensor

- We'll form a wonderful object known as the **Maxwell Field Tensor** ("tensor" because it has 2 indices, μ and ν)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$A^\mu = \left(\frac{V}{c}, A_x, A_y, A_z\right)$$

$$\partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- "If in doubt, write it out"

$$\begin{array}{c} \\ \mu = 0 \\ \mu = 1 \\ \mu = 2 \\ \mu = 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} \nu = 0 & \nu = 1 & \nu = 2 & \nu = 3 \end{array} \\
\begin{pmatrix}
\partial^0 A^0 - \partial^0 A^0 & \partial^0 A^1 - \partial^1 A^0 & \partial^0 A^2 - \partial^2 A^0 & \partial^0 A^3 - \partial^3 A^0 \\
\partial^1 A^0 - \partial^0 A^1 & \partial^1 A^1 - \partial^1 A^1 & \partial^1 A^2 - \partial^2 A^1 & \partial^1 A^3 - \partial^3 A^1 \\
\partial^2 A^0 - \partial^0 A^2 & \partial^2 A^1 - \partial^1 A^2 & \partial^2 A^2 - \partial^2 A^2 & \partial^2 A^3 - \partial^3 A^2 \\
\partial^3 A^0 - \partial^0 A^3 & \partial^3 A^1 - \partial^1 A^3 & \partial^3 A^2 - \partial^2 A^3 & \partial^3 A^3 - \partial^3 A^3
\end{pmatrix}
\end{array}$$

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$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$A^{\mu} = \left(\frac{V}{c}, A_x, A_y, A_z\right)$$

$$\partial^{\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

- Multiplying this out,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell's equations

Maxwell's equations …

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{0} \qquad \vec{\nabla}\times\vec{B} - \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

… can be nicely written as

$$\partial_\mu F^{\mu\nu} = -\mu_0\, J^\nu$$

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$$

Transformation of EM fields

- We can use the Maxwell Field Tensor to deduce how electromagnetic fields transform between frames:

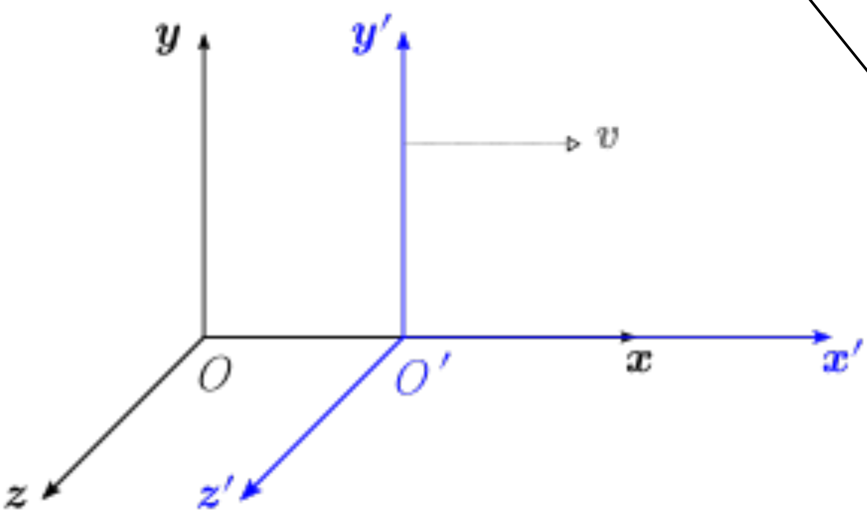
$$F'^{\mu\nu} = L^{\mu}{}_{\kappa}\, L^{\nu}{}_{\lambda}\, F^{\kappa\lambda}$$

Maxwell field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Lorentz transformation

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -v\gamma/c & 0 & 0 \\ -v\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Transformation of EM fields

- Multiplying out each component of this equation:

$$E_x{}' = E_x \qquad B_x{}' = B_x$$

$$E_y{}' = \gamma\left(E_y - vB_z\right) \qquad B_y{}' = \gamma\left(B_y + vE_z/c^2\right)$$

$$E_z{}' = \gamma\left(E_z + vB_y\right) \qquad B_z{}' = \gamma\left(B_z - vE_y/c^2\right)$$

- $\vec{E}$ and $\vec{B}$ “intermingle” under a Lorentz transformation!

Another path to relativity?

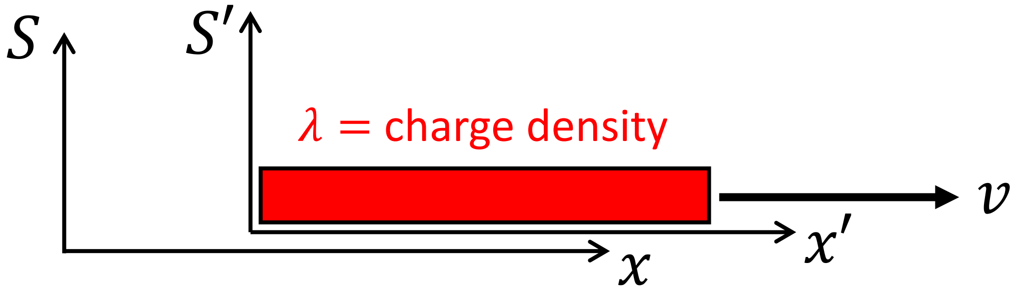
- **Maxwell's Equations keep the same mathematical form under a Lorentz transformation – but not under a Galilean transformation**
- This was already known by Lorentz in the 1890s, before Einstein published Special Relativity
- In the modern story Einstein's postulates are presented first, but it can all be derived from electromagnetism!

H.Lorentz (1853-1928)

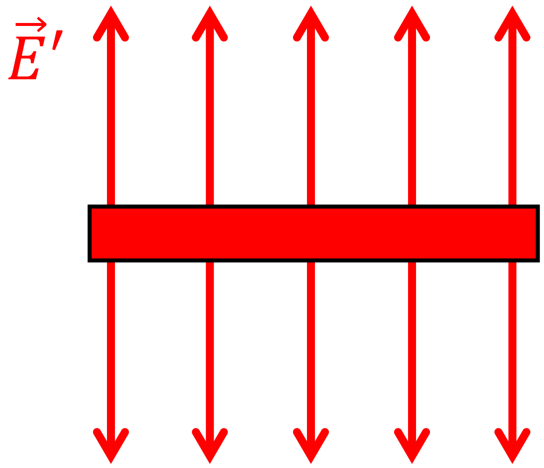
https://www.nobelprize.org/nobel_prizes/physics/laureates/1902/lorentz-bio.html

Magnetic field of a current

- When a line of charge moves, it becomes a current



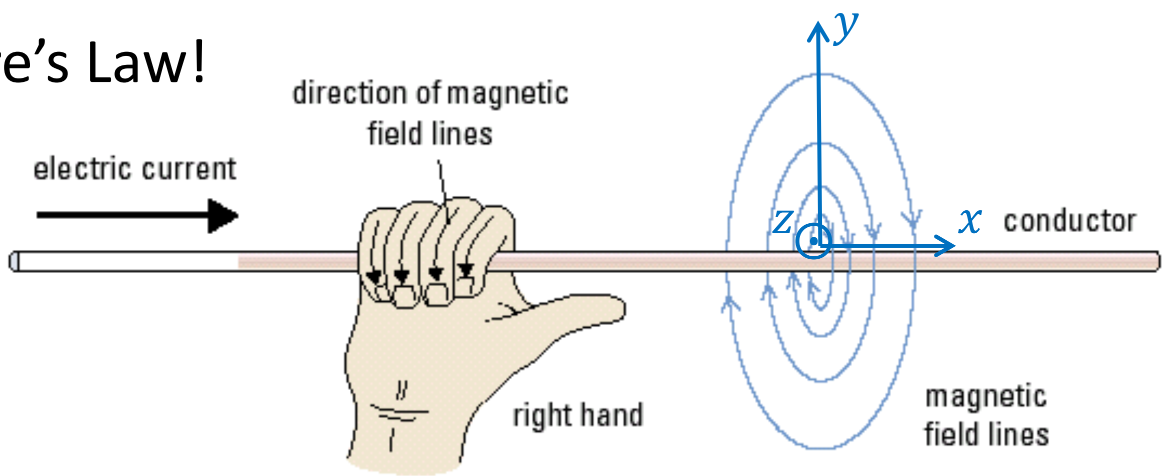
- First consider the field of the stationary charge in S':



Gauss's Law: $E'_y = \dfrac{\lambda'}{2\pi\varepsilon_0 y'}$ $\vec{B}' = \vec{0}$

Magnetic field of a current

- *What is the magnetic field in S?*
- Lorentz transformation: $B_z = \frac{\gamma v E_y{}'}{c^2} = \frac{\gamma v \lambda'}{2\pi\varepsilon_0 y' c^2}$
- Length contraction implies that the line density of charge in S is $\gamma\lambda'$, and the current in S is then $I = \gamma\lambda' v$
- Hence, using $y' = y$ and $c^2 = \frac{1}{\varepsilon_0\mu_0}$, we find $B_z = \frac{\mu_0 I}{2\pi y}$
- Ampere's Law!



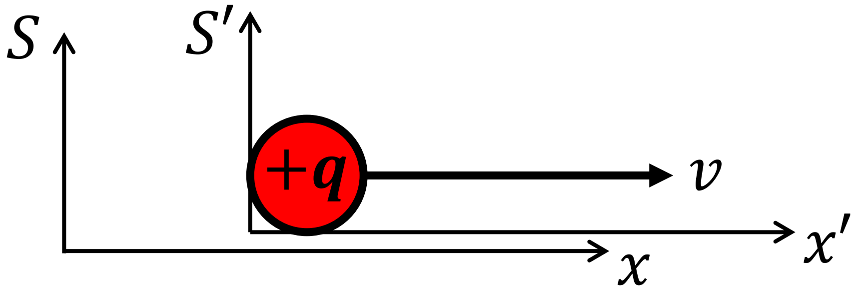
https://www.quora.com/Does-a-current-carrying-conductor-act-as-a-magnet-Does-it-have-to-be-wound-around-an-iron-core-to-get-the-magnetic-effect

Transformation of EM fields

- What are the implications of this formulation of electromagnetism?
- **Magnetism is not an independent phenomenon, but a relativistic effect** that occurs when electrostatics are observed in a different frame
- If the speed of light were infinite, there would be no magnetism!

Field of a moving charge

- What is the electric field of a charge $+q$ moving through S?



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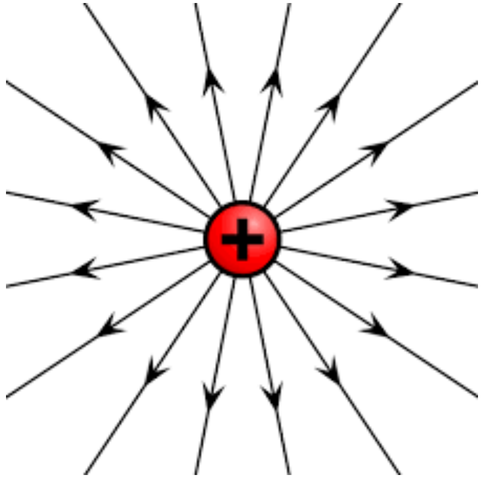
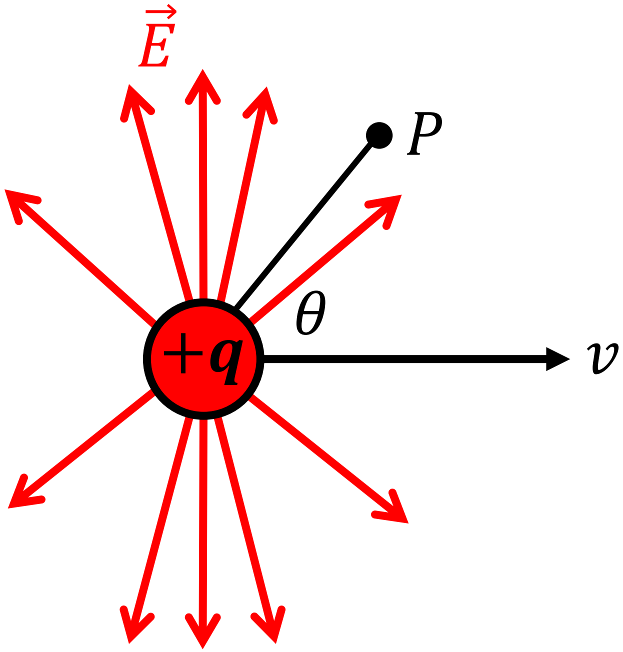


$$\vec{E}' = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{r}' \qquad \vec{B}' = \vec{0}$$

https://courses.lumenlearning.com/boundless-physics/chapter/the-electric-field-revisited/

Field of a moving charge

- Lorentz transformations: $\vec{E} = \left(E_x{}', \gamma E_y{}', \gamma E_z{}'\right)$
- Evaluate the field at $t = 0$: $\vec{r}\,' = (\gamma x, y, z)$
- After some algebra: $r'^2 = \gamma^2 r^2[1 - (v \sin\theta\, / c)^2]$



$$\vec{E} = \frac{q\,\vec{r}}{4\pi\varepsilon_0 \gamma^2 r^3 [1 - (v \sin\theta\, / c)^2]^{3/2}}$$

EM energy-momentum tensor

- The energy-momentum tensor for electromagnetism is:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\lambda}\, F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu}\, F_{\kappa\lambda}\, F^{\kappa\lambda}\right)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- How do we de-code this expression??
- $F^{\mu}{}_{\lambda} = \eta_{\lambda\nu}\, F^{\mu\nu} = \sum_{\nu} \eta_{\lambda\nu} F^{\mu\nu}$
- $F_{\kappa\lambda} = \eta_{\kappa\mu}\, \eta_{\lambda\nu}\, F^{\mu\nu}$
- $F_{\kappa\lambda}\, F^{\kappa\lambda} = \sum_{\kappa}\sum_{\lambda} F_{\kappa\lambda}\, F^{\kappa\lambda}$

EM energy-momentum tensor

- The energy density is given by the first component, T^{00}:

$$T^{00} = \frac{1}{\mu_0}\left(F^0{}_\lambda\, F^{0\lambda} - \frac{1}{4}\eta^{00}\, F_{\kappa\lambda}\, F^{\kappa\lambda}\right)$$

- Evaluating this expression, $T^{00} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$
- The flux of energy in the i-direction is given by T^{0i}:

$$T^{0i} = \frac{1}{\mu_0}\left(F^0{}_\lambda\, F^{i\lambda} - \cancel{\frac{1}{4}\eta^{0i}\, F_{\kappa\lambda}\, F^{\kappa\lambda}}\right)$$

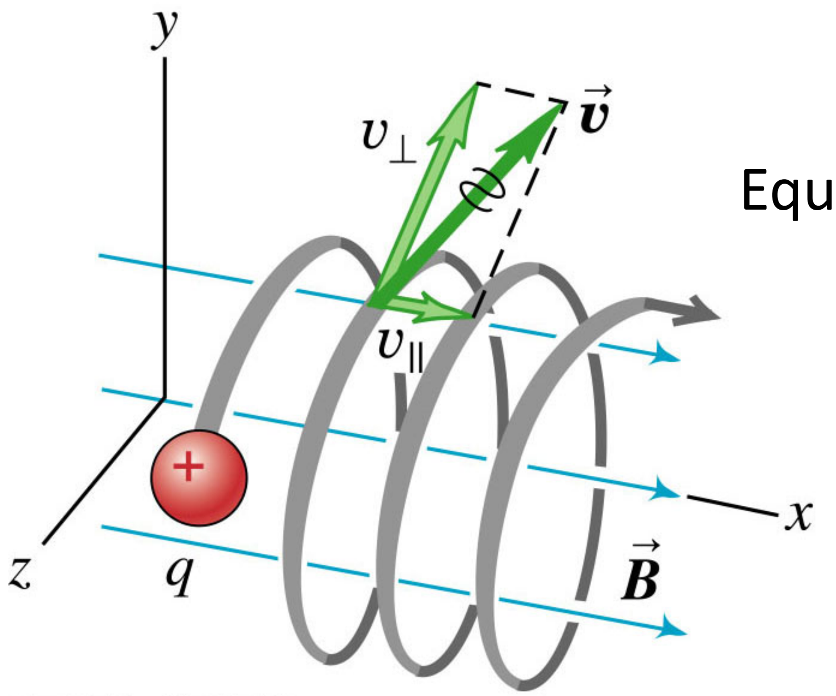
$$= 0$$

- Evaluating this expression, $T^{0i} = \frac{1}{\mu_0}\frac{(\vec{E}\times\vec{B})}{c}$

Force on a moving charge

- The Lorentz force law can be expressed in index notation as

$$\frac{dp^\mu}{d\tau} = q\, F^\mu{}_\nu\, u^\nu$$



Equivalent to $\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$

Gauges

- In electrostatics, the **zero of potential is arbitrary**
- In general, *if* $A^{\mu} \rightarrow A^{\mu} + \partial^{\mu} f$, *where* f *is any function of* $(\vec{x}, t)$, *the electromagnetic fields* $\vec{E}$ *and* $\vec{B}$ *are unchanged!* (We can show this by substituting in $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$)
- This freedom is known as a **gauge choice** – it's convenient to choose f such that $\partial_{\mu} A^{\mu} = 0$
- That's because we then find, $\partial_{\mu} F^{\mu\nu} = \partial_{\mu} \partial^{\mu} A^{\nu}$
- Combining with Maxwell's Equation $\partial_{\mu} F^{\mu\nu} = -\mu_0 J^{\nu}$:

$$\boxed{\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A^{\mu} = -\mu_0\, J^{\mu}}$$