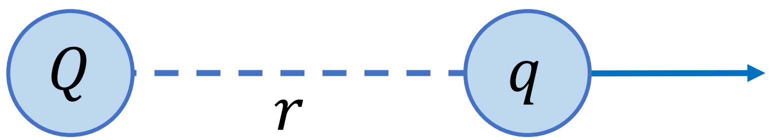
# Class 12: Summary of GR

*In this class we will summarize the conceptual principles of General Relativity, linking them to the mathematics of space-time geometry*

# Why is gravity special?

- **Gravitational mass is equal to inertial mass**

- Consider two charges interacting **electrostatically**:




$$F = \frac{kQq}{r^2} = m_I a$$

(inertial mass $m_I$)

- Consider two masses interacting **gravitationally**:

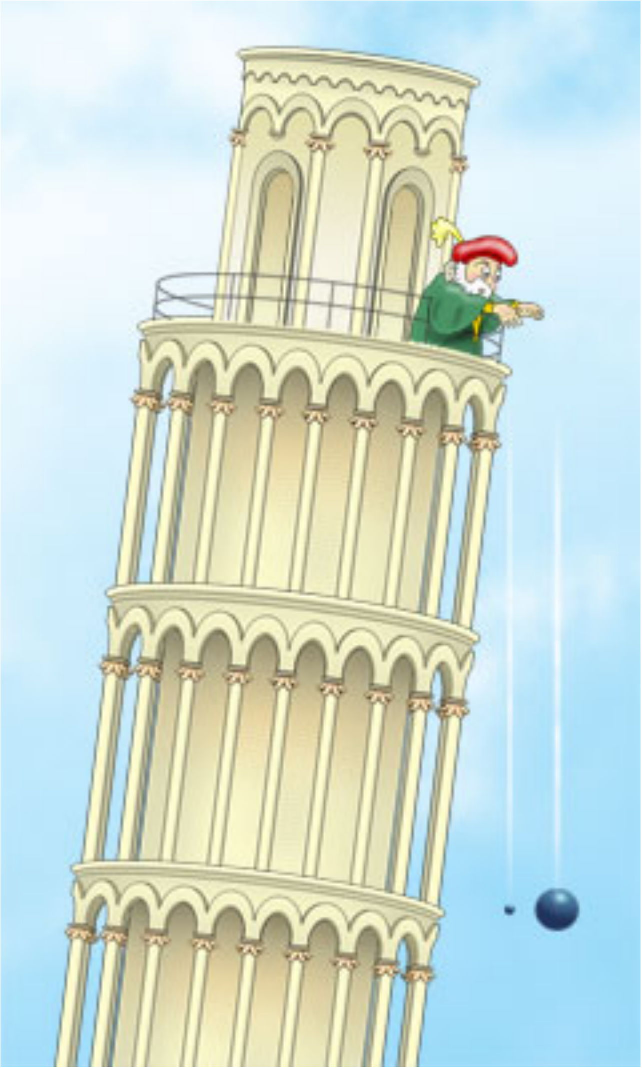

$$F = \frac{GMm_G}{r^2} = m_I a$$

(inertial mass $m_I$)

- Remarkably, $m_I$ equals $m_G$ to 13 significant figures!

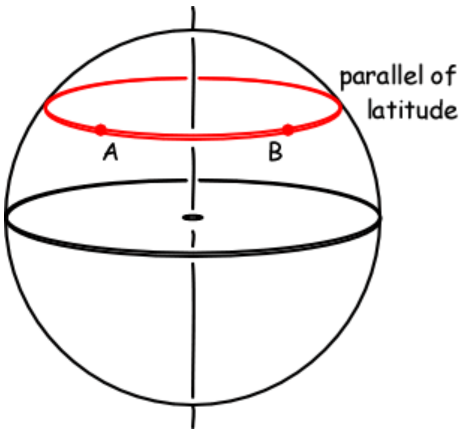
# Why is gravity special?

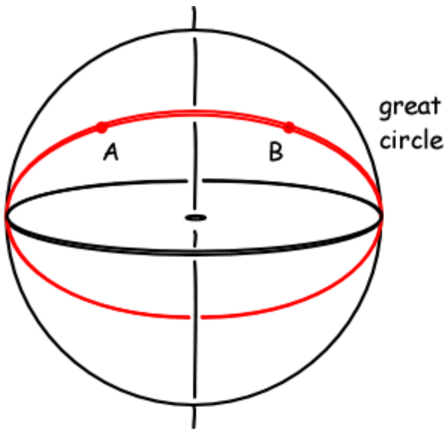
- All objects experience the **same acceleration** in a gravitational field
- If launched with the same position/velocity, all objects follow the **same trajectory** in a gravitational field
- *What property of empty space determines the trajectory of all objects?*

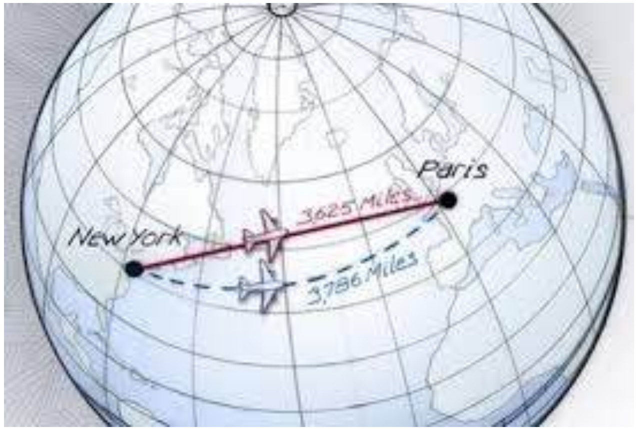


# Geodesics

- In a general space, a unique trajectory between two points can be defined by the shortest path between the points, known as a **geodesic**



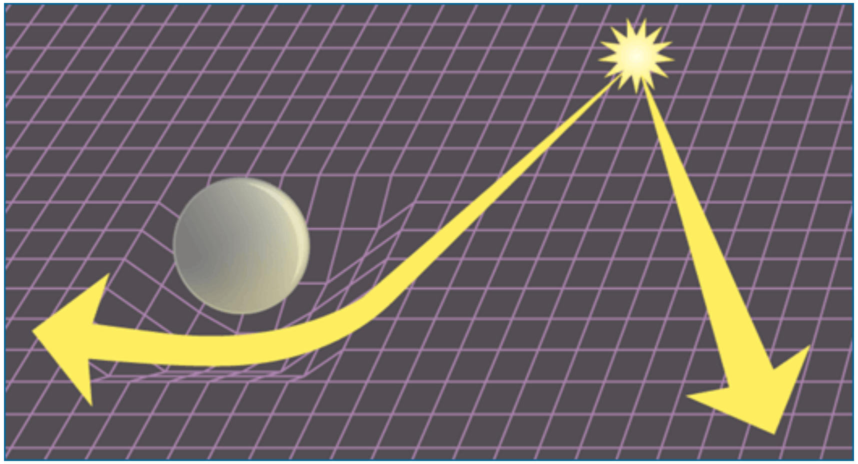



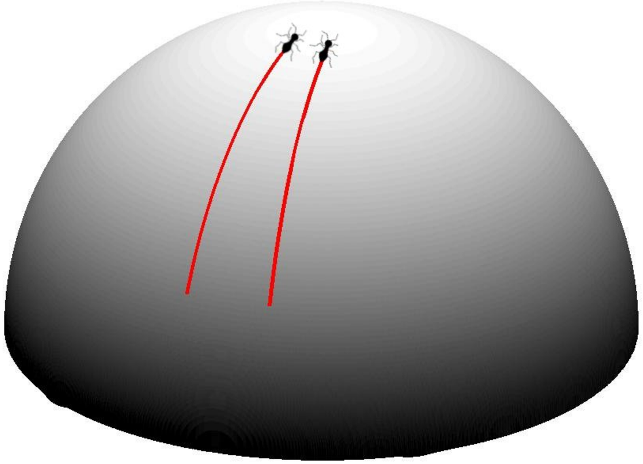




- The geometric characteristics of a space define unique geodesic paths in that space

# Geodesics

- The geodesic hypothesis of General Relativity states that **a free particle follows a geodesic in space-time**

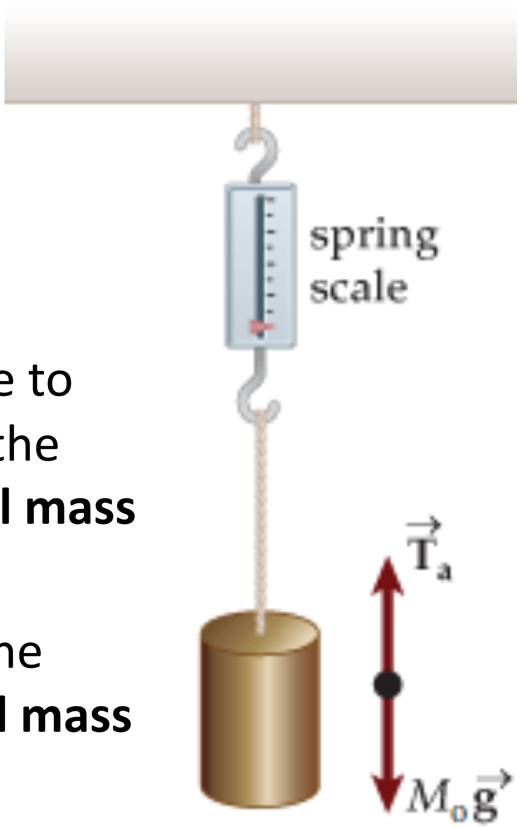




- The **gravitational field shapes space-time**, specifying the paths that particles must follow
- Note that this **applies to 4D space-time, not 3D space** (throwing a ball and a light ray across a room looks different)

# Geodesics

- The geodesic hypothesis explains why gravitational and inertial mass are the same thing




- If I drop an object at the Earth's surface, it accelerates downward at $9.8\ m/s^2$, following a geodesic

- To hold the object at rest, I must apply an upward force to accelerate it at $9.8\ m/s^2$ relative to this geodesic – in the Newtonian view, this force is **determined by its inertial mass**

- In the Newtonian view, the required force is equal to the weight of the object as **determined by its gravitational mass**

- ***In the view of General Relativity, measuring gravitational weight is the same as measuring resistance to acceleration***

# The Equivalence Principle

- Newton said: "an object remains at rest or in uniform motion unless acted on by a force"

- This law applies in **inertial frames** – in other co-ordinate frames we feel "fictitious forces"

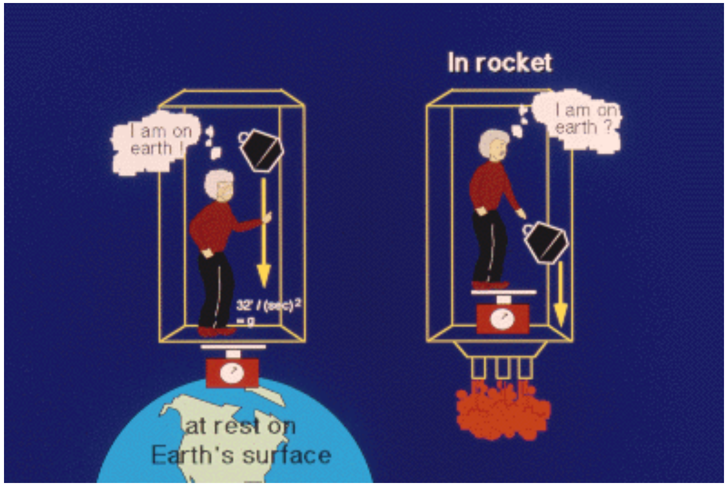
- If gravity no longer counts as a force (because it's produced by space-time), then **a frame at rest on the surface of the Earth is no longer an inertial frame**

# The Equivalence Principle

- Einstein formulated this idea as the Equivalence Principles …


- **A freely-falling frame is locally equivalent to an inertial frame**

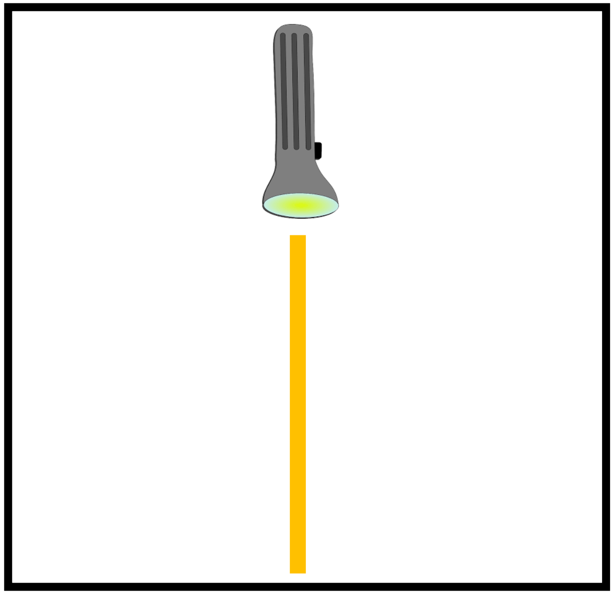
- **A gravitational field is locally equivalent to an acceleration**




- "Equivalent" means that *performing any experiment would yield the same results*

# The Equivalence Principle

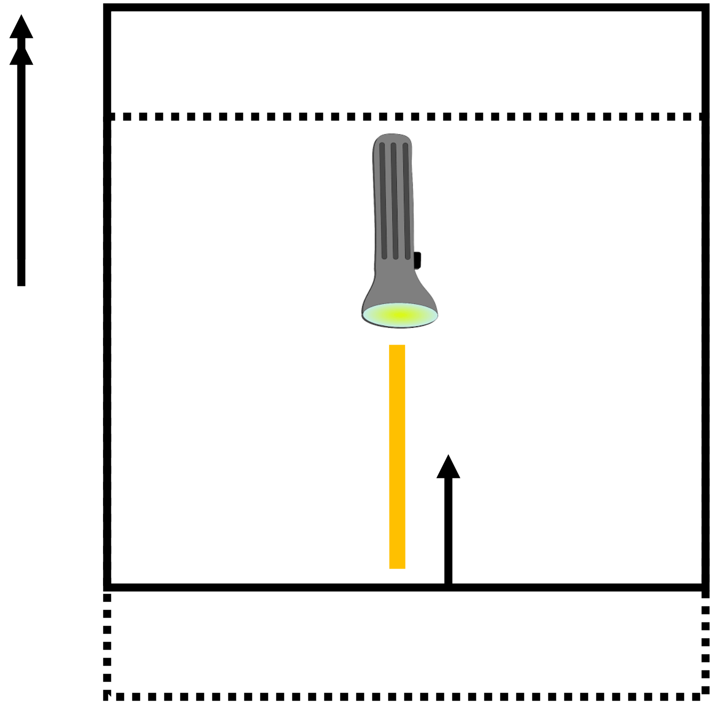
- The Equivalence Principle is directly testable …



- Consider a laboratory on the surface of the Earth
- Shine a ray of light from the ceiling to the floor
- Measure the frequency of the light ray at the floor
- If the laboratory is an **inertial frame**, the frequency would be unchanged

# The Equivalence Principle

- The Equivalence Principle is directly testable …



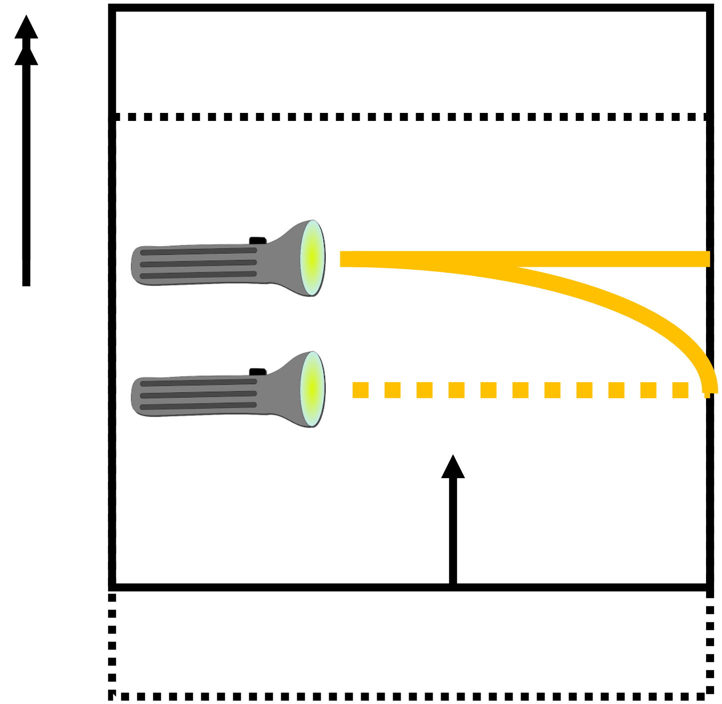
- By the Equivalence Principle, the experiment is the same as if the laboratory were **accelerating upward**
- By the time the light ray reaches the floor, the floor has a relative upward speed
- By the Doppler effect, the frequency of the light would change (→ **non-inertial frame**)

# The Equivalence Principle

- We can carry out this experiment, and the frequency of light changes as it moves through a gravitational field
- Since frequency is like a clock, this implies that **time runs differently at each point in a gravitational field**
- The presence of gravity in a frame implies that it is non-inertial

# The Equivalence Principle

- The Equivalence Principle is consistent with the notion that space-time is “curved”

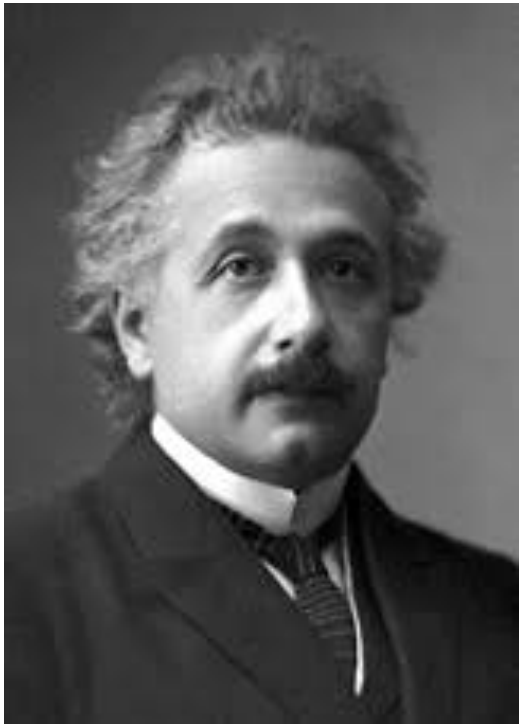


- Back in the laboratory on the surface of the Earth, shine the light ray horizontally
- The result is the same as if the laboratory were accelerating upward
- In that view, the **light ray would appear to bend**

# The Einstein Equation

- The final piece is to predict **how a gravitating body affects the curvature of space-time**
- Einstein's famous equation links the curvature of space-time to the matter-energy within it …

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



# The space-time metric

- With these concepts in place, we can now summarize the mathematical detail

- The geometrical properties of the curved space-time $x^\mu$ are described by the **space-time metric:**

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

- For example, the proper time intervals $d\tau$ ticked by a freely-falling clock is related to the interval $dt$ when it's at rest in a frame containing gravity by $d\tau = \sqrt{-g_{tt}}\, dt$

- In a weak gravitational field, $g_{tt}(\vec{x}) = -1 - 2\phi/c^2$

# The space-time metric

- The space-time metric **far from any gravitating mass** is the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

- The space-time metric **in the empty space around a black hole** is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)(c\,dt)^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2[d\theta^2 + (\sin\theta\;d\phi)^2]$$

- The space-time metric **of a homogeneous/isotropic Universe** is the FRW metric:

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2)\right]$$

# Geodesics

- Freely-falling objects follow **geodesics** $x^\mu(\tau)$ of the space-time, which are *completely specified by the space-time metric*:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$$

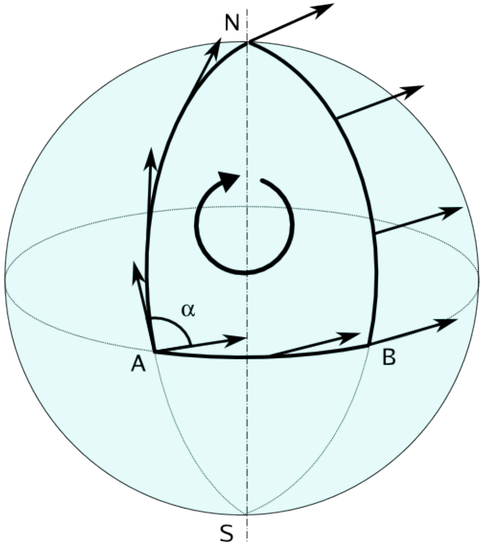
- Geodesics are paths which extremize the proper time for an object to move between two points

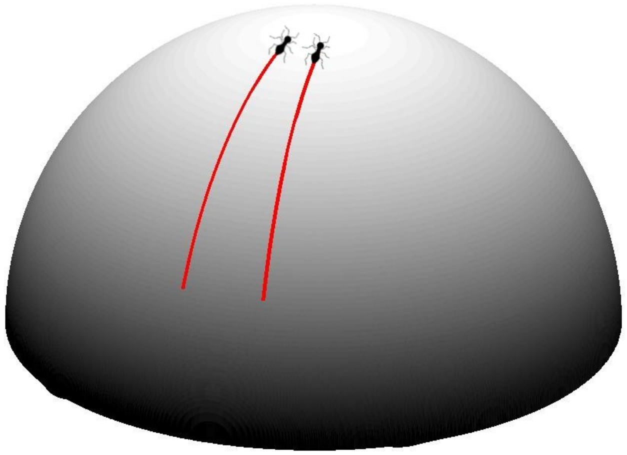
# Space-time curvature

- The **curvature** of space-time is described by the **Riemann tensor**, which is again completely specified by the metric:

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}_{\lambda\nu} - \partial_{\nu}\Gamma^{\kappa}_{\lambda\mu} + \Gamma^{\kappa}_{\mu\alpha}\Gamma^{\alpha}_{\lambda\nu} - \Gamma^{\kappa}_{\nu\alpha}\Gamma^{\alpha}_{\lambda\mu}$$

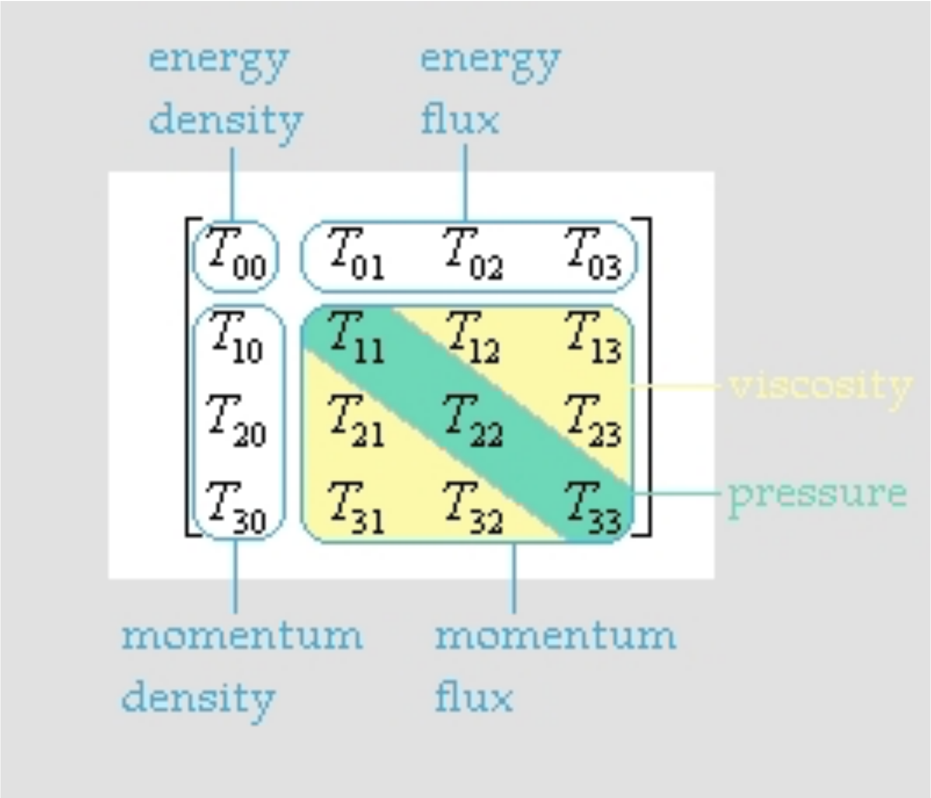
- The Riemann tensor describes the rotation of a vector during parallel transport, or the divergence of geodesics



# Matter and energy

- The distribution/flows of matter and energy are described by the **energy-momentum tensor** $T^{\mu\nu}$

# Einstein's Equation

- **Einstein's Equation** relates space-time curvature to the energy-momentum tensor

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Here, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the **Einstein tensor**, which is again *completely specific by the space-time metric*
- $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ is the **Ricci tensor** (average of the Riemann tensor)
- $R$ is the **Ricci scalar**, $R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$

# Summary of GR

*Space-time tells matter how to move*
(geodesic equation)

*Matter tells space-time how to curve*
(Einstein's equation)