Class 11: Cosmology

In this class we will explore how General Relativity can be applied to the expansion of the Universe, linking its development from the Big Bang to the matter-energy it contains

Class 11: Cosmology

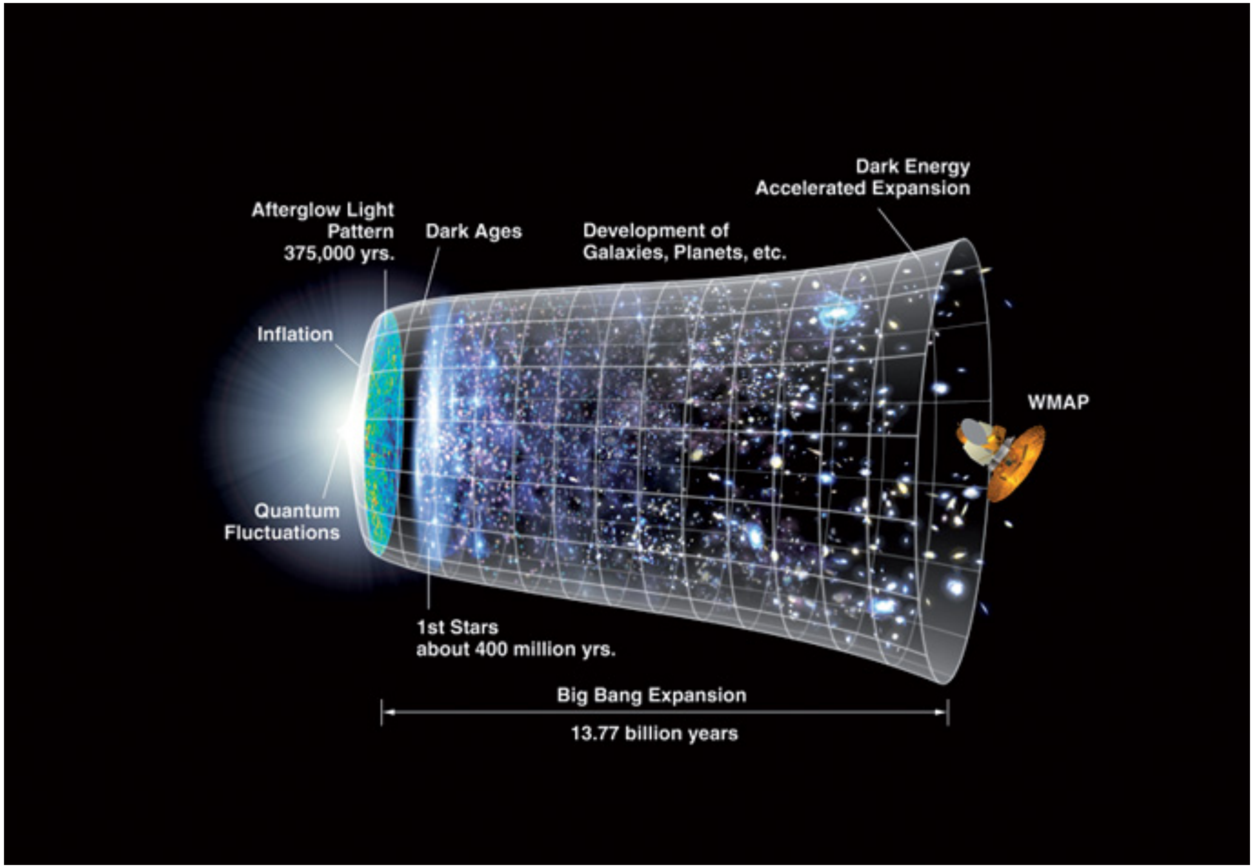
At the end of this session you should be able to …

- … understand how the **cosmological principle** leads to the space-time metric of a homogeneous and isotropic Universe
- … use geodesics of light rays to show that **light is redshifted** as it propagates in an expanding Universe
- … describe different definitions of **cosmological distance**
- … link the expansion of the Universe to its **matter-energy content** and **curvature** using Einstein's equation
- … calculate the **look-back time** and **distance** to a galaxy

The cosmological principle

- There is another situation in which the equations of GR can be solved: **cosmological observations of the Universe**

The Universe is expanding from a Hot Big Bang!

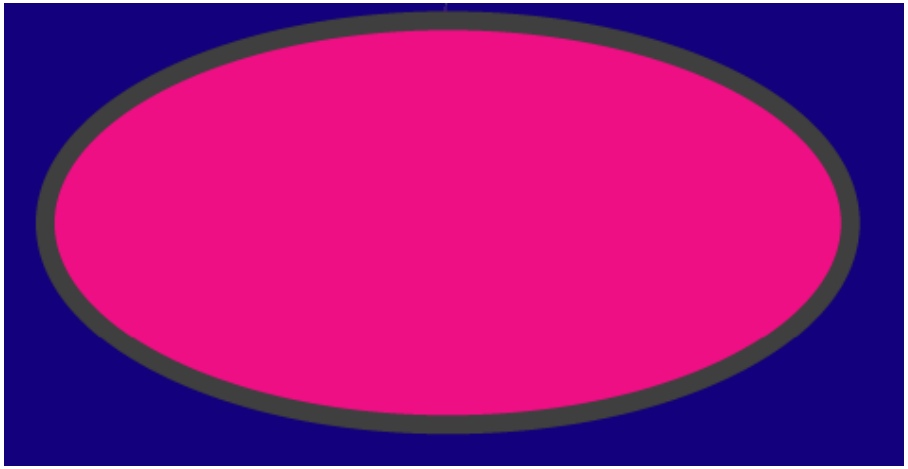


https://en.wikipedia.org/wiki/Chronology_of_the_universe

The cosmological principle

- Although the Universe is irregular in detail, we can “smooth out” its contents into a **set of space-filling particles** which are uniformly expanding in all directions

http://wise2.ipac.caltech.edu/staff/jarrett/ngss/wise_LSS.html

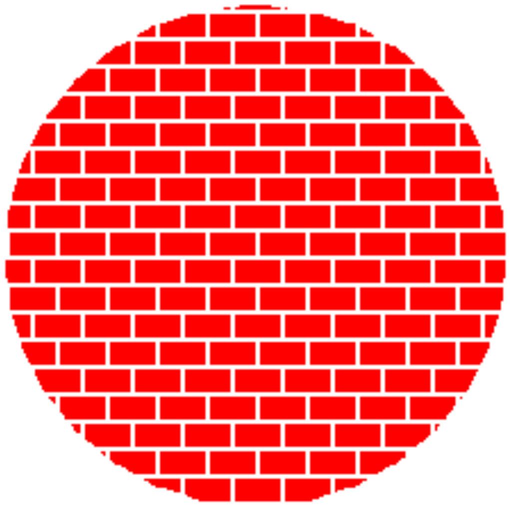


http://background.uchicago.edu/~whu/intermediate/map2.html

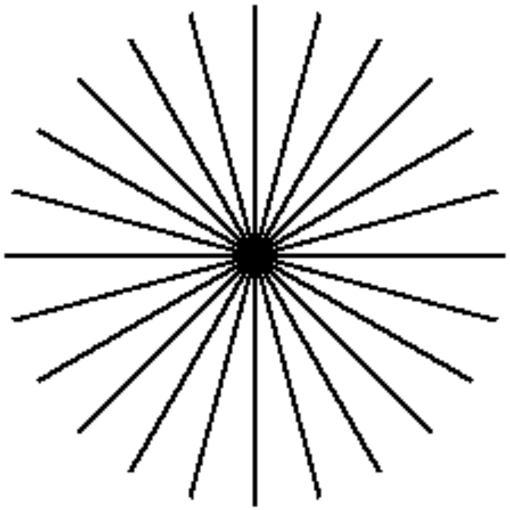
- These are the *assumptions of homogeneity and isotropy*, together known as the **cosmological principle**

The cosmological principle

- **Homogeneity** means that the properties of the Universe are the *same in every location*
- **Isotropy** means that the properties of the Universe are the *same in every direction*



Homogeneous but not isotropic



Isotropic but not homogeneous

http://www.astro.ucla.edu/~wright/cosmo_01.htm

The cosmological principle

- **Why is this such a powerful assumption?**
- Suppose each of these space-filling particles carries a "fundamental observer" – by homogeneity, the experience of each of these observers is identical
- Any part of the Universe is representative of the whole – **homogeneous Universes can be studied locally**
- There is an **absolute cosmic time** – the proper time for each fundamental observer – for which the Universe itself acts as the synchronization agent

The metric of the Universe

- What do these symmetries imply about the space-time metric of the expanding Universe? It must take the form …

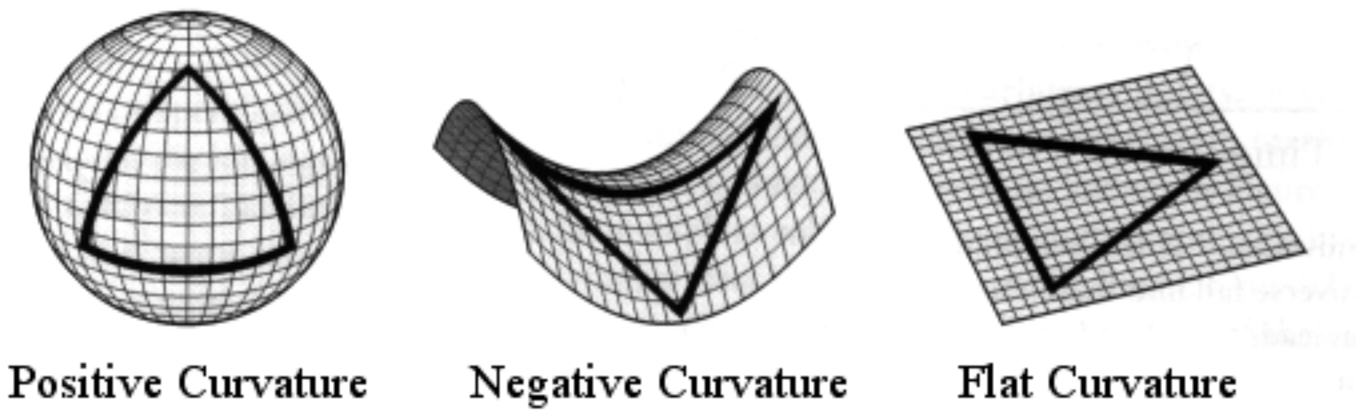
$$ds^2 = -c^2 dt^2 + a(t)^2 dl^2$$

This piece because cosmic time t is just proper time τ for fundamental observers (which have $dl = 0$)

This piece ensures that the Universe remains homogeneous and isotropic as it expands, with all distances simply scaling as $a(t)$, the cosmic scale factor

The metric of the Universe

- Now let's think about the form of the proper separation dl
- Homogeneity and isotropy imply that **the curvature of space must everywhere be equal and independent of orientation**, so can be written as a single number K
- The curvature can be flat ($K = 0$), positive or negative



http://abyss.uoregon.edu/~js/cosmo/lectures/lec15.html

The metric of the Universe

- We can derive the form of dl by analogy with the 2D surface of a sphere embedded in a 3D Euclidean space, which satisfies the equation $x^2 + y^2 + z^2 = 1/K$
- For a constant curvature 3D surface embedded in a 4D Euclidean space: $x^2 + y^2 + z^2 + w^2 = 1/K$
- In the 4D Euclidean space: $dl^2 = dx^2 + dy^2 + dz^2 + dw^2$
- Transform (x, y, z) to spherical polar co-ordinates (r, θ, ϕ)
- From above: $w^2 = 1/K - r^2$, hence $dw^2 = \frac{r^2\, dr^2}{1/K - r^2}$
- Putting it all together: $dl^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + (\sin\theta\, d\phi)^2)$

The metric of the Universe

- The assumptions of homogeneity and isotropy, allowing for constant curvature K, produce the **Robertson-Walker metric**

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2)\right]$$

- The co-ordinates (r, θ, ϕ) are fixed for “fundamental observers” who measure proper time t

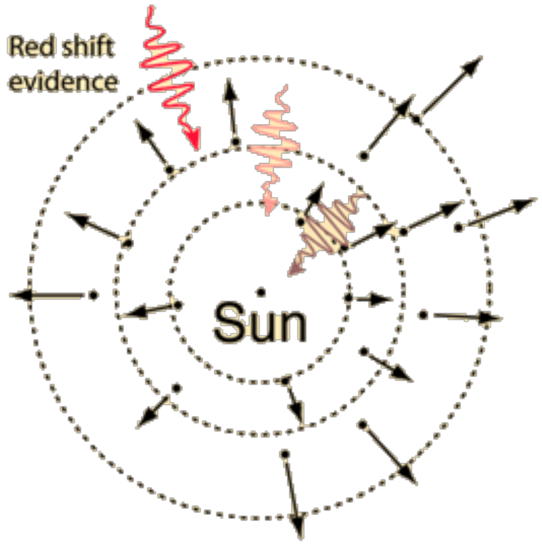
- These are also known as **co-moving observers**, since they expand with the Universe

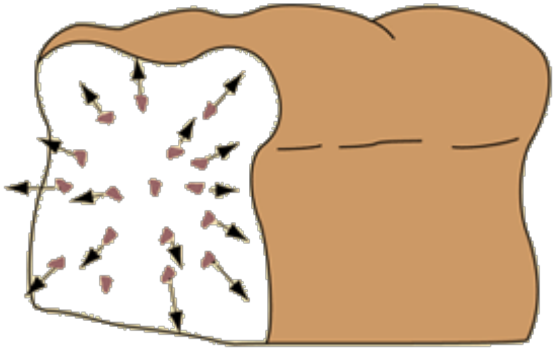
- The non-zero metric elements are hence $g_{tt} = -1$, $g_{rr} = \frac{a(t)^2}{1-Kr^2}$, $g_{\theta\theta} = a(t)^2 r^2$ and $g_{\phi\phi} = a(t)^2 r^2 (\sin\theta)^2$

Hubble's Law

- The proper distance L between any 2 fundamental observers, separated by co-ordinate distance l, increases as $L(t) = a(t)\, l$
- The rate of increase of this distance is $\frac{dL}{dt} = \frac{da}{dt}\, l = \frac{da/dt}{a(t)}\, L$
- Hence the **speed of recession is proportional to distance**

This relation is known as Hubble's Law – the coefficient of proportionality is Hubble's constant, $H = \dot{a}/a$



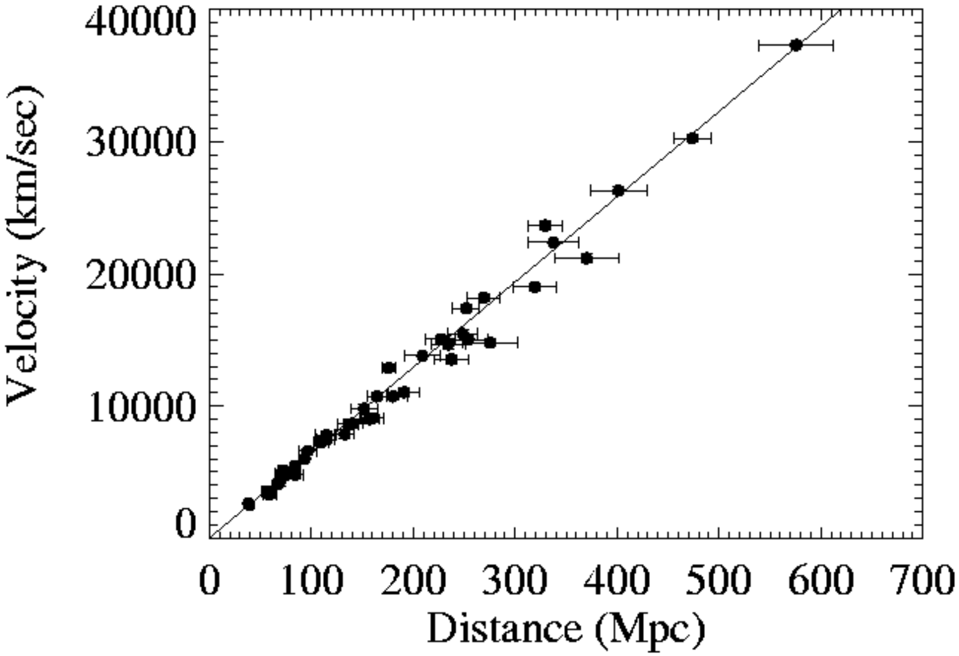


Every raisin in a rising loaf of raisin bread will see every other raisin expanding away from it.

http://hyperphysics.phy-astr.gsu.edu/hbase/Astro/hubble.html

Hubble's Law

- Hubble's Law has been beautifully confirmed by measuring distances and recession velocities of nearby galaxies:



https://ned.ipac.caltech.edu/level5/Tyson/Tyson2.html

- The velocities are deduced from *Doppler shift in spectral lines*

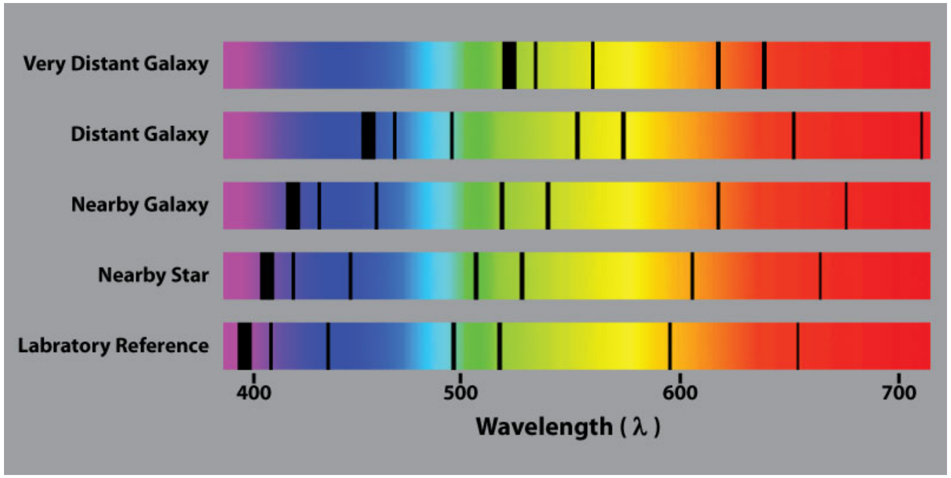
Redshifting

- The frequency ω of a light ray, travelling in expanding space with scale factor $a(t)$, changes such that $\omega \propto 1/a$

- This behaviour is called **redshifting of light**, where redshift z is defined as the ratio of emission/observation frequencies

$$1 + z = \frac{\omega_{emitted}}{\omega_{observed}} = \frac{a_{observed}}{a_{emitted}}$$

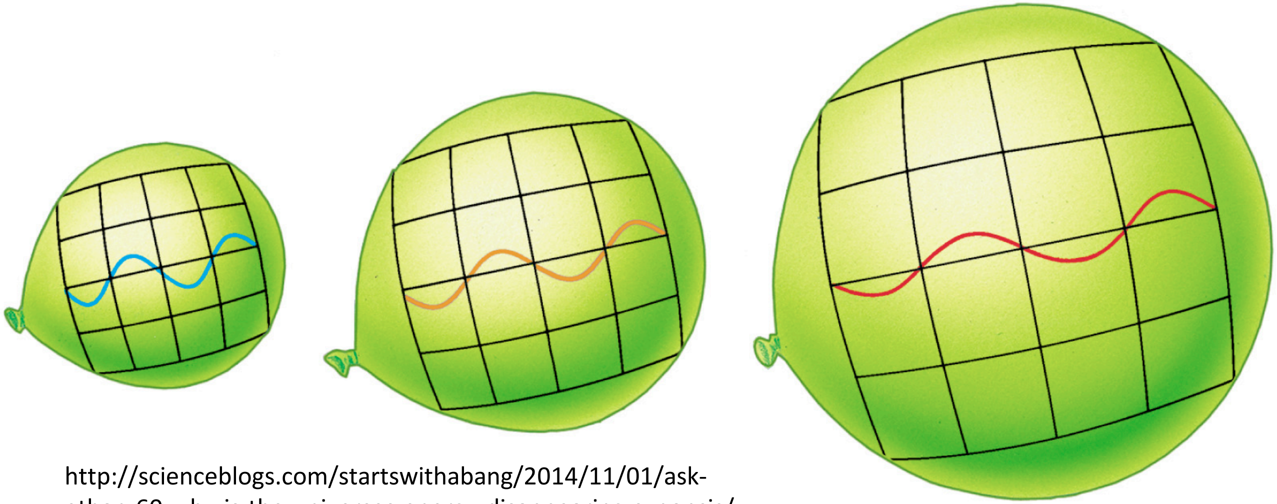
Redshift of light from distant galaxies can be measured from the frequencies of spectral lines



http://www.space-exploratorium.com/doppler-shift.htm

Redshifting

- In a crude analogy, the wavelength of light is stretching with the expansion of the Universe

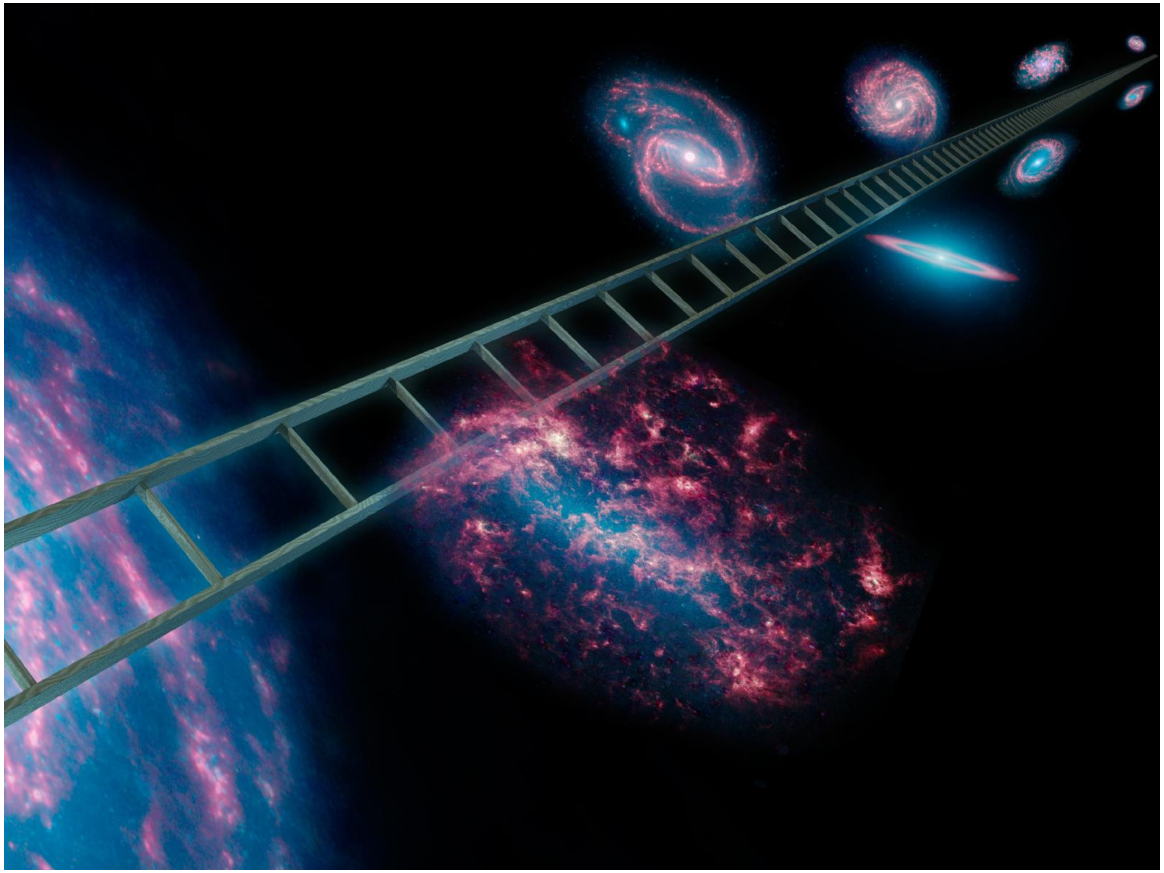


http://scienceblogs.com/startswithabang/2014/11/01/ask-ethan-60-why-is-the-universes-energy-disappearing-synopsis/

- More accurately, we can think of the light *undergoing many small Doppler shifts* between pairs of fundamental observers, or *travelling along a geodesic in expanding space-time*

Distances in expanding space

- Now let's consider how to measure **distances in the expanding Universe**, from $r = 0$ to a galaxy at co-ordinate r



https://www.nasa.gov/mission_pages/spitzer/multimedia/pia15818.html

Distances in expanding space

- Now let's consider how to measure **distances in the expanding Universe**, from $r = 0$ to a galaxy at co-ordinate r

- The **proper distance** L is the distance measured if, at the same cosmic time t, a chain of fundamental observers to the galaxy added up their infinitesimal proper separations

- $dL = \sqrt{g_{rr}}\, dr = \frac{a(t)\, dr}{\sqrt{1-Kr^2}}$ (since $d\theta = d\phi = 0$)

- Integrating:

$$L = a(t) \sin^{-1}\left(r\sqrt{K}\right)/\sqrt{K} \qquad K > 0$$

$$L = a(t)\, r \qquad K = 0$$

$$L = a(t)\ \sinh^{-1}\left(r\sqrt{-K}\right)/\sqrt{-K} \qquad K < 0$$

Distances in expanding space

- The proper distance is not very useful in observational cosmology, since we can't measure it! More useful measures are the **luminosity distance** and **angular diameter distance**

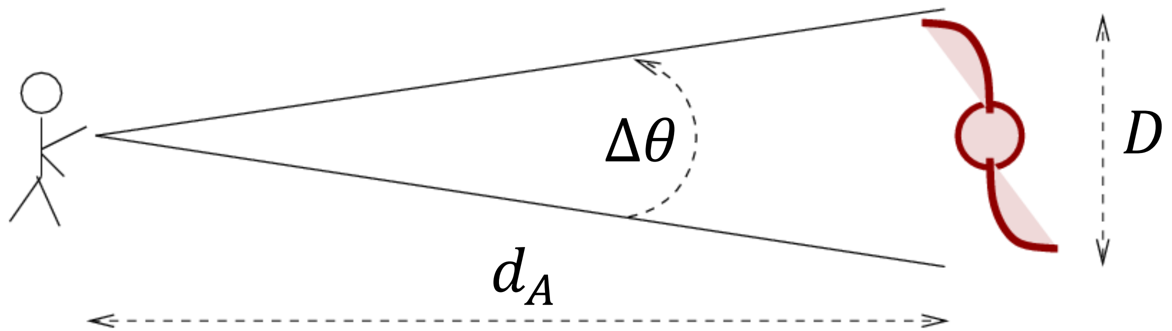
Luminosity distance

Angular diameter distance

https://www.nasa.gov/mission_pages/galex/pia14095.html

Distances in expanding space

- **Angular diameter distance**: a light source at co-ordinate r has proper diameter D and apparent angular size $\Delta\theta$



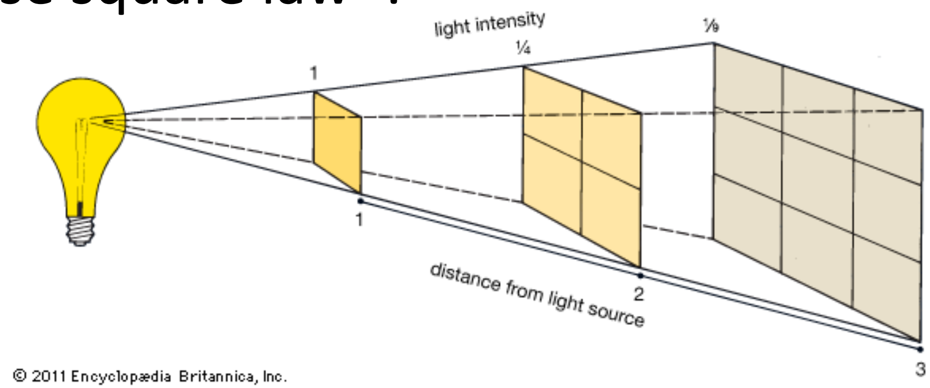
- The angular diameter distance is defined as: $d_A = D/\Delta\theta$

- From the metric: $D = \Delta L = a(t)\, r\, \Delta\theta$

- Hence $d_A = a(t_{em})\, r$, where t_{em} is the light emission time

Distances in expanding space

- **Luminosity distance**: consider photons emitted by a distant galaxy, travelling to our telescopes. What is the equivalent of the "inverse square law"?



- Since frequency $\omega \propto 1/a$, photons *lose energy* as they travel

- The *co-ordinate time interval changes* between the photons

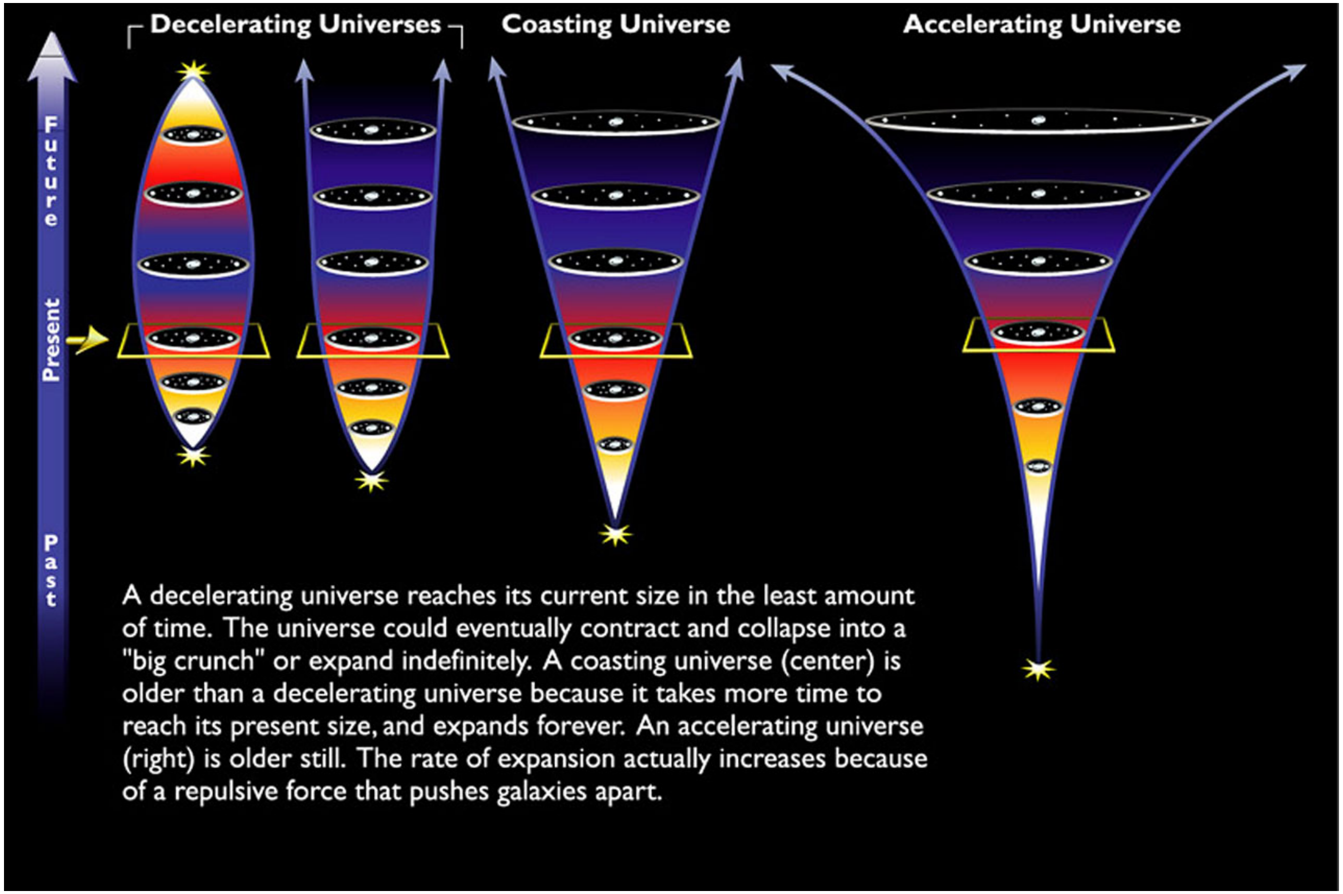
- **Why?** Since $ds = 0$, we know that $\int_{t_{em}}^{t_{obs}} \frac{dt}{a(t)} = \frac{1}{c^2} \int_{r_{em}}^{0} \frac{dr}{\sqrt{1-Kr^2}}$

Distances in expanding space

- So, $\int_{t_{em}}^{t_{obs}} \frac{dt}{a(t)} =$ constant $= \int_{t_{em}+\delta t_{em}}^{t_{obs}+\delta t_{obs}} \frac{dt}{a(t)}$ for the 2nd photon
- This implies that $\frac{\delta t_{obs}}{a(t_{obs})} = \frac{\delta t_{em}}{a(t_{em})}$
- **Each photon decreases in energy by $\boldsymbol{a(t_{em})/a(t_{obs})}$, and the time between them increases by the same factor**
- The flux of energy received is $f = \frac{L\, a(t_{em})^2}{4\pi r^2}$
- The **luminosity distance** is defined by $d_L = \sqrt{\frac{L}{4\pi f}} = \frac{r}{a(t_{em})}$

Matter-energy in the Universe

- We now relate the expansion of the Universe to its **matter-energy content**



https://www.spacetelescope.org/images/opo9919k/

Matter-energy in the Universe

- This is done via Einstein's equation from the previous class:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- The metric $g_{\mu\nu}$ is given by

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + (\sin\theta)^2 d\phi^2) \right]$$

- We won't go through the algebra of computing the Ricci tensor $R_{\mu\nu}$ from $g_{\mu\nu}$, but the non-zero components are:

$$R_{tt} = -\frac{3}{c^2} \frac{\ddot{a}}{a} \qquad R_{ii} = \left[\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{2Kc^2}{a^2} \right] \frac{g_{ii}}{c^2}$$

Matter-energy in the Universe

- What is the **energy-momentum tensor** of the homogeneous and isotropic Universe? We will consider 2 components …

- First, the **smoothly distributed particles filling the Universe** contain an energy density, but no pressure, such that

$$T_{00} = \rho(t)\, c^2 \qquad\qquad T_{others} = 0$$

- Second, it turns out that **empty space contains an energy density** (the "cosmological constant") that is required to describe our observations

$$T_{\mu\nu} = \Lambda\, g_{\mu\nu}$$

The Friedmann equation

- We can use Einstein's Equation to derive the **Friedmann Equation**, which describes the expansion of the Universe in terms of its matter-energy content:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho(t)}{3} - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Expansion rate (where $\dot{a}/a$ is the Hubble parameter)

Effect of matter

Effect of curvature

Effect of cosmological constant

Density parameters

- We **normalize the scale factor such that today, $\boldsymbol{a = 1}$**, with today's density as ρ_0 and Hubble parameter as H_0

- The Friedmann equation today: ${H_0}^2 = \frac{8\pi G \rho_0}{3} - Kc^2 + \frac{\Lambda c^2}{3}$

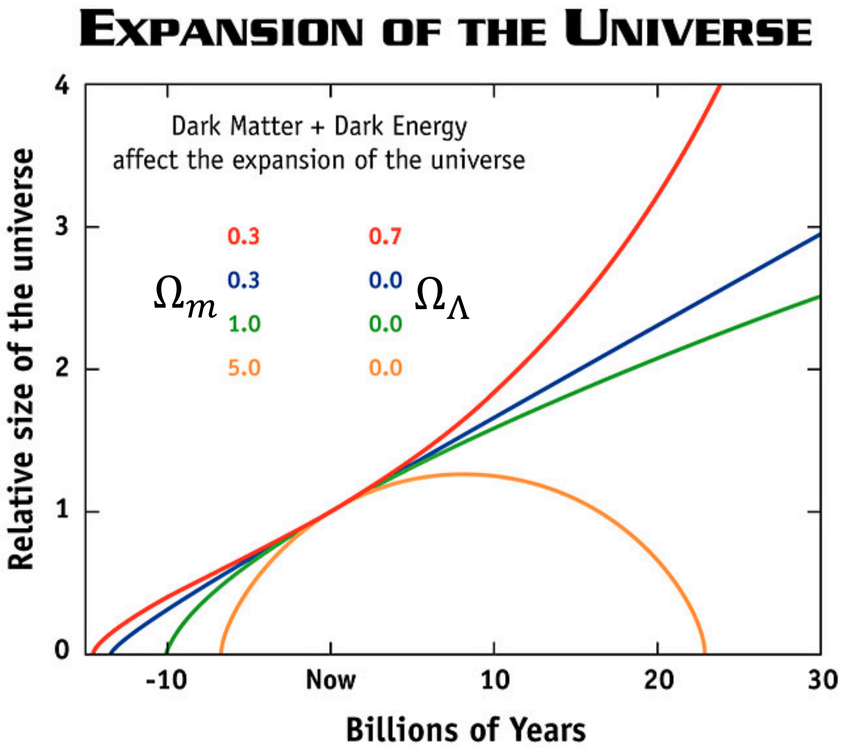
- It's convenient to define the **critical density** $\rho_{crit} = \frac{3{H_0}^2}{8\pi G}$ and the dimensionless density parameters

$$\Omega_m = \frac{\rho}{\rho_{crit}} = \frac{8\pi G \rho}{3{H_0}^2} \qquad \Omega_K = -\frac{Kc^2}{{H_0}^2} \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3{H_0}^2}$$

- These are hence related conveniently by $\Omega_m + \Omega_K + \Omega_\Lambda = 1$

Density parameters

- We also note that as the Universe expands, the *matter density dilutes*: $\rho(t) = \rho_0/a^3$ – the Λ density doesn't!
- The Friedmann equation is then: $\frac{1}{{H_0}^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_K}{a^2} + \Omega_\Lambda$



The expansion history of the Universe depends on the values of Ω_m and Ω_Λ (with $\Omega_K = 1 - \Omega_m - \Omega_\Lambda$)

http://www.preposterousuniverse.com/blog/2006/01/26/the-future-of-the-universe/