Class 10: Einstein Equation

In this class we will introduce Einstein's equations of General Relativity, which provide the all-important link between space-time curvature and matter-energy

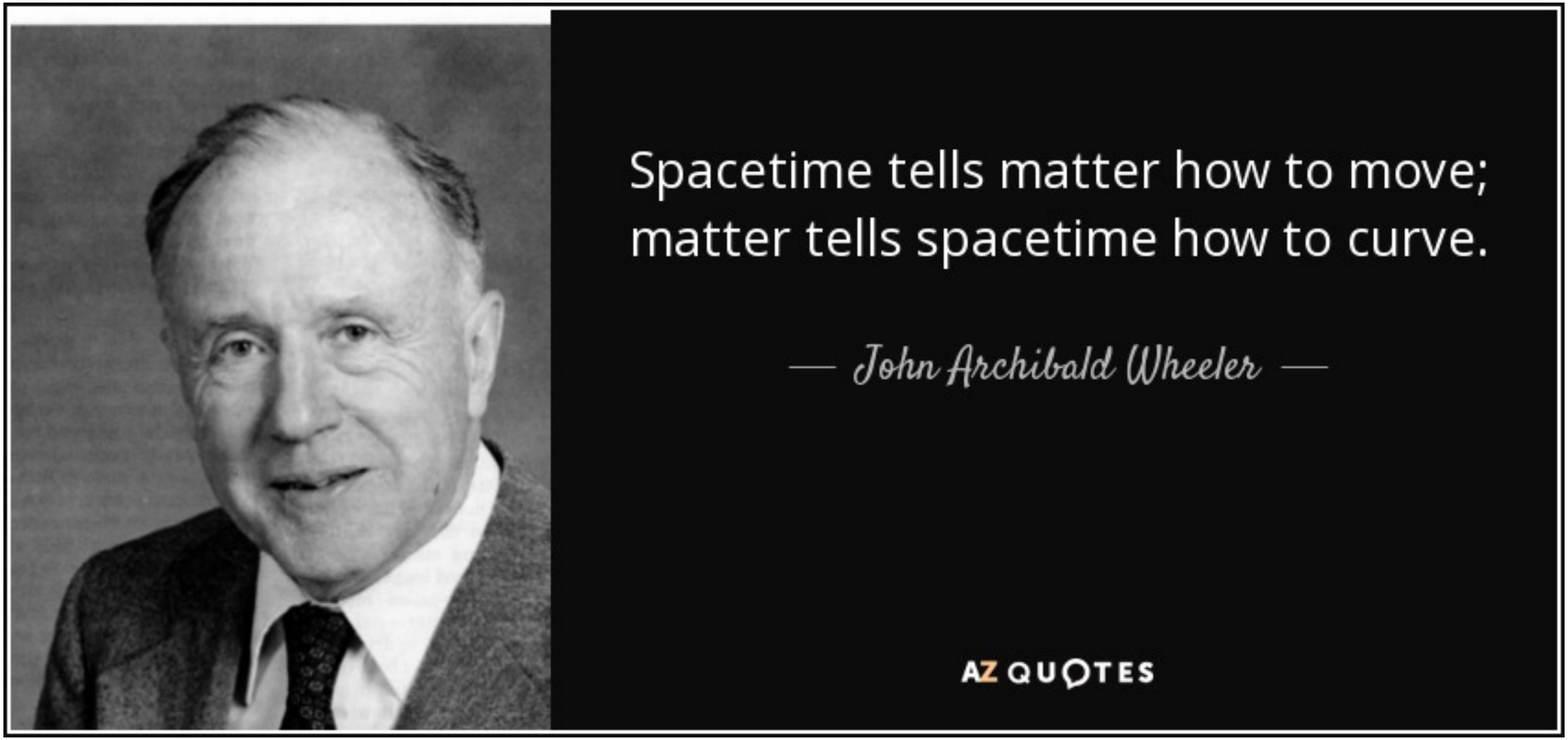
Class 10: Einstein Equation

At the end of this session you should be able to …

- … recognize **Einstein's equations of General Relativity**, which link space-time curvature to matter-energy
- … be familiar with the components of the Einstein equations – the **Ricci tensor** $R_{\mu\nu}$ and the **Ricci scalar** R
- … show that Einstein's equations **recover Newton's laws** in the appropriate weak-field limit
- … understand how, mathematically, we can **test whether or not we are in empty space**, free of matter

Linking curvature and matter

John Wheeler has a famous summary of General Relativity:

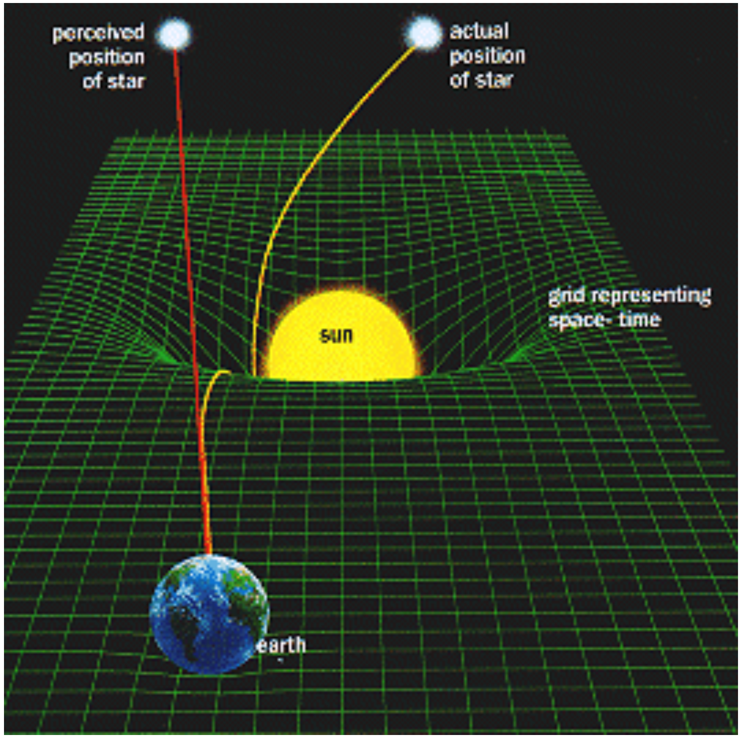


http://www.azquotes.com/quote/712153

Linking curvature and matter

Space-time tells matter how to move
(geodesic equation)

Matter tells space-time how to curve
(Einstein's equation)



https://www.quora.com/What-is-the-best-way-to-imagine-space-time

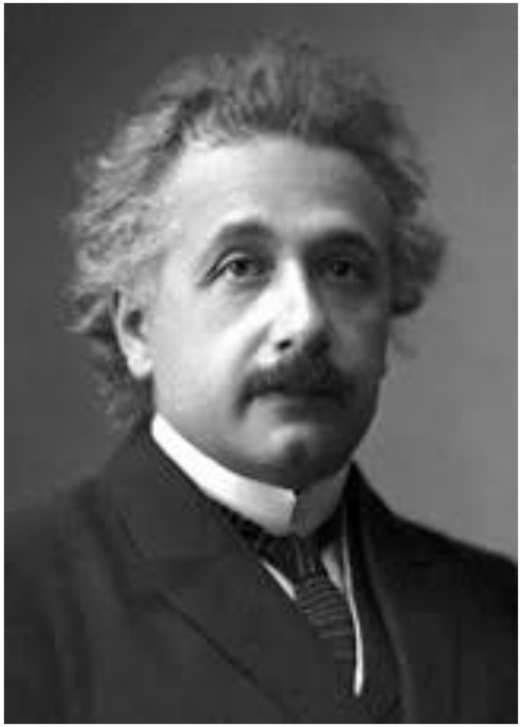
Ricci tensor

- We have seen that general space-time curvature is described by the **Riemann tensor** $R^{\kappa}_{\mu\lambda\nu}$ (4 indices!)
- However, the metric of space-time, and its energy-momentum content, are described by the simpler objects $g_{\mu\nu}$ and $T_{\mu\nu}$ (2 indices!)
- We can produce an object of corresponding complexity from the Riemann tensor by contracting it as follows: $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$
- This is known as the **Ricci tensor**, $R_{\mu\nu}$ – it is an *"average of the Riemann tensor"*
- **Einstein proposed his famous equation linking $\boldsymbol{R_{\mu\nu}}$ and $\boldsymbol{T_{\mu\nu}}$**

Einstein equation

- Einstein proposed that:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



https://www.nobelprize.org/nobel_prizes/physics/laureates/1921/einstein-bio.html

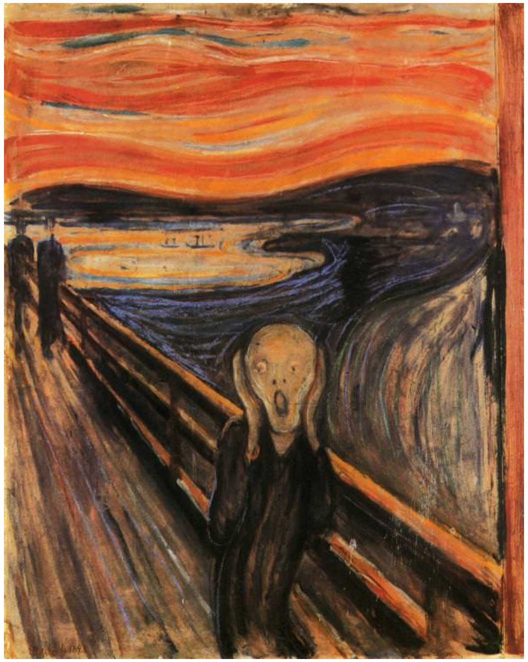
- $G_{\mu\nu}$ is called the **Einstein tensor**
- It is related to the Ricci tensor $R_{\mu\nu}$ by $\boldsymbol{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}$
- R is the **Ricci scalar,** $R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$

Einstein equation

Matter tells space-time how to curve
(Einstein's equation)

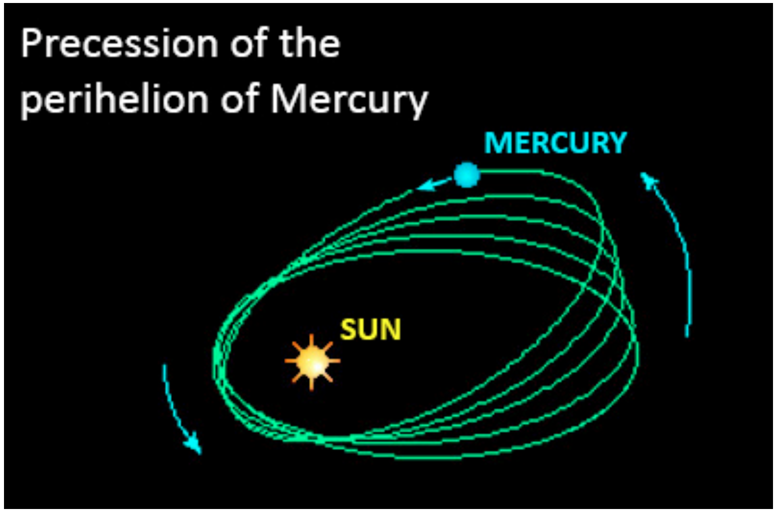
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- This is "10 equations in one"! (μ and ν can each take on 4 values, but the equation is symmetric, $T_{\mu\nu} = T_{\nu\mu}$)
- Each equation is a non-linear differential equation for $g_{\mu\nu}$

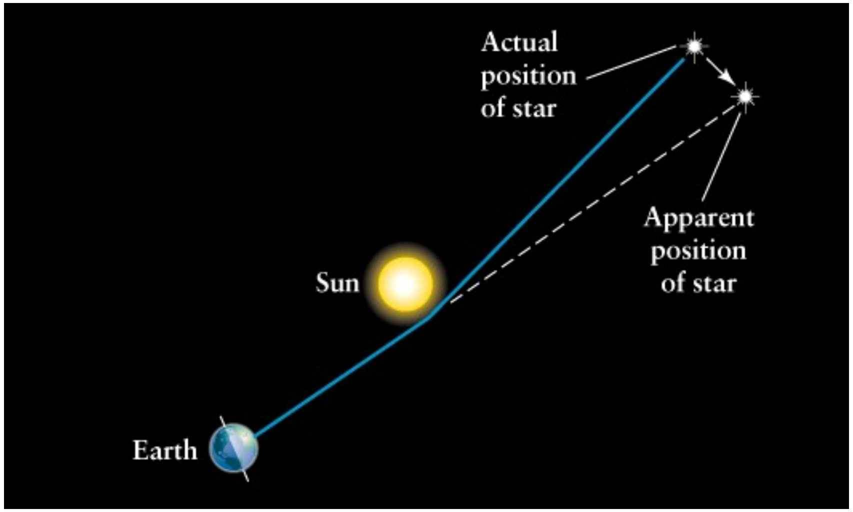


Einstein equation

- Why Einstein's equation?? First, it is **mathematically consistent**
- Second, it **reproduces Newtonian gravity in the weak-field limit**
- Third, it makes **predictions which have been experimentally verified** – e.g. the *perihelion of Mercury, gravitational deflection*



http://www.eniscuola.net/en/mediateca/precession-of-mercury/



http://andybohn.com/research/lensing.html

Einstein equation

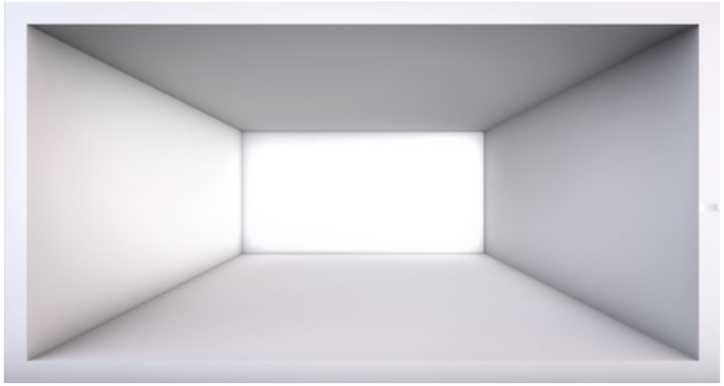
- The Einstein equation is $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
- If we multiply both sides by $g^{\lambda\mu}$ and perform a little algebra, we find a useful re-write: $R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, T_{\lambda}^{\lambda}\right)$
- We can also show after some algebra that …

$$R_{\mu\nu} = \partial_{\lambda}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\kappa}_{\kappa\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\kappa}\Gamma^{\kappa}_{\mu\lambda}$$

- **There's no need to memorize this!** But the point is that the *Christoffel symbols* (which determine the geodesics), and the *Ricci tensor* (which links to the matter-energy), are all **completely determined by the metric** (after heavy calculating!)

What is “empty space”?

- **How can we tell that we are in empty space (free of matter)?**



- It would wrong to say that $g_{\mu\nu} = 0$ in empty space, of course
- $\Gamma^{\lambda}_{\mu\nu} \neq 0$ – empty space can be curved by the matter outside it!
- Einstein’s equations show us that $\boldsymbol{R_{\mu\nu} = 0}$ **in empty space**
- For example, to verify the Schwarzschild metric we used in the previous class, we would need to show that $R_{\mu\nu} = 0$