Class 6: Curved Space and Metrics

In this class we will discuss the meaning of spatial curvature, how distances in a curved space can be measured using a metric, and how this is connected to gravity

Class 6: Curved Space and Metrics

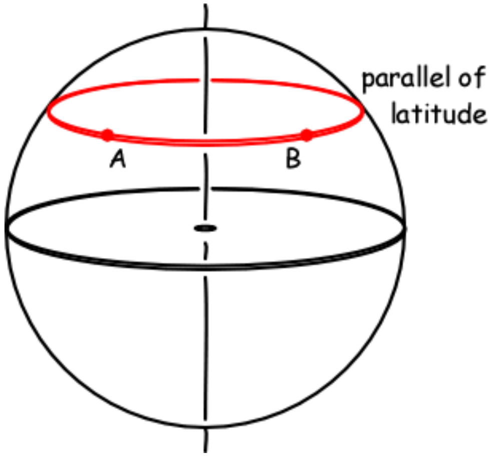
At the end of this session you should be able to …

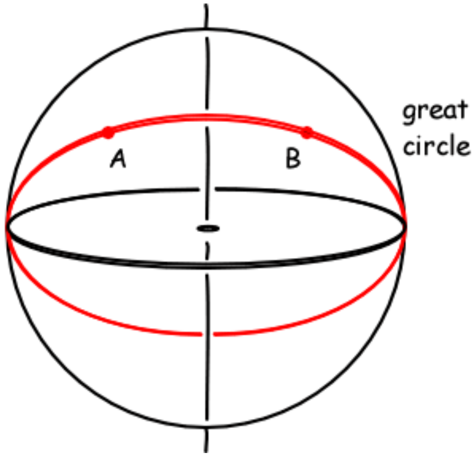
- … describe the **geometrical properties of curved spaces** compared to flat spaces, and how observers can determine whether or not their space is curved
- … know how the **metric of a space** is defined, and how the metric can be used to compute distances and areas
- ... make the connection between **space-time curvature and gravity**
- … apply the **space-time metric** for Special Relativity and General Relativity to determine proper times and distances

Properties of curved spaces

- In the last Class we discussed that, according to the Equivalence Principle, objects ***"move in straight lines in a curved space-time"***, in the presence of a gravitational field

- So, what is a straight line on a curved surface? We can define it as *the shortest distance between 2 points*, which mathematicians call a **geodesic**

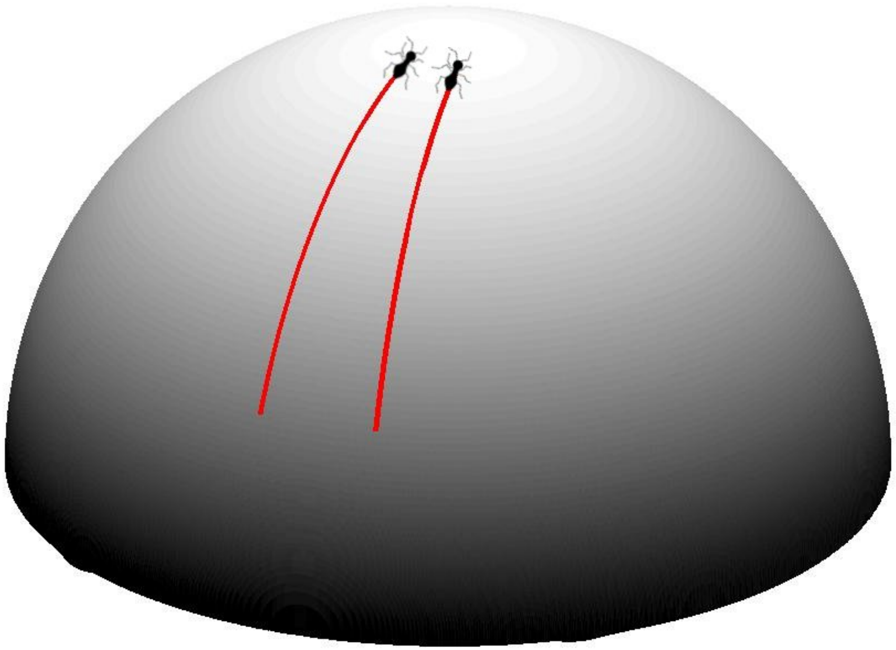




https://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/non_Euclid_curved/index.html
https://www.quora.com/What-are-the-reasons-that-flight-paths-especially-for-long-haul-flights-are-seen-as-curves-rather-than-straight-lines-on-a-screen-Is-map-distortion-the-only-reason-Or-do-flight-paths-consider-the-rotation-of-the-Earth

Properties of curved spaces

Equivalently, a geodesic is a path that would be travelled by an ant walking straight ahead on the surface!



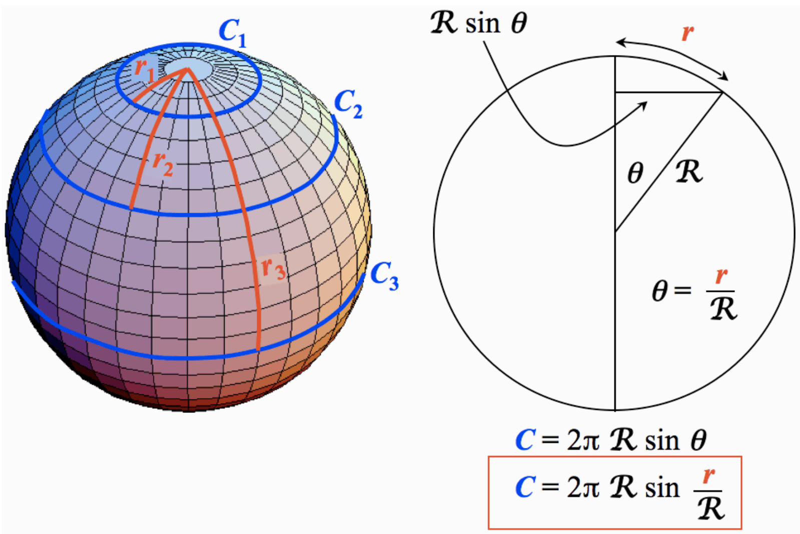
http://astronomy.nmsu.edu/geas/lectures/lecture28/slide03.html

- Consider two ants starting from different points on the Equator, both walking North
- These geodesics are both “straight lines”, but they are converging
- **Parallel lines converge or diverge on a curved surface**

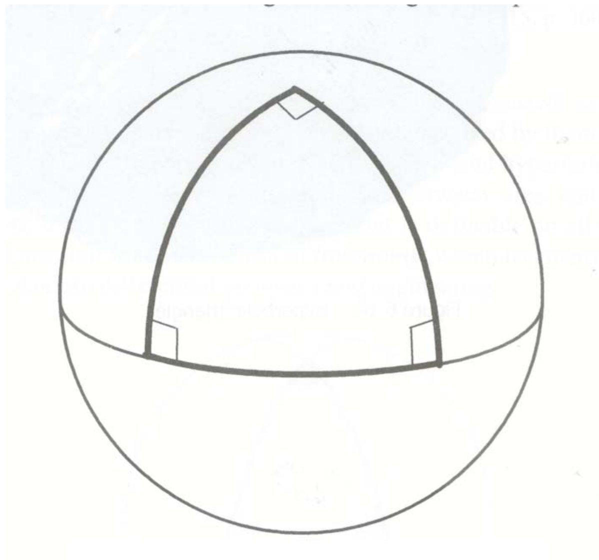
Properties of curved spaces

Some other counter-intuitive properties of curved surfaces:

- The circumference of a circle of radius r is not $2\pi r$
- The area of a circle of radius r is not πr^2
- The angles of a triangle do not add up to $180°$



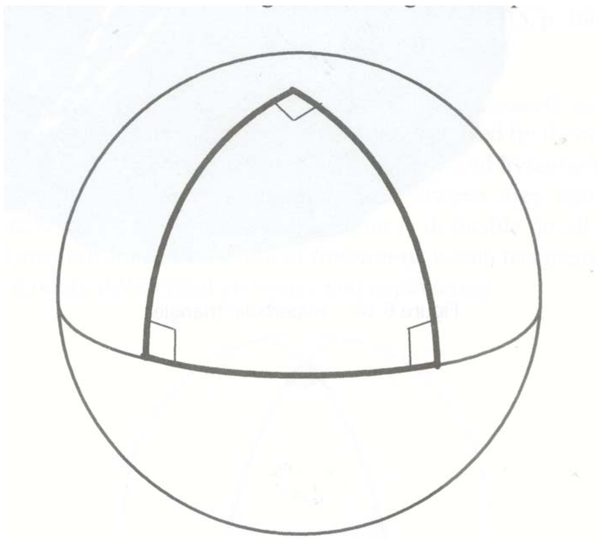
http://people.virginia.edu/~dmw8f/astr5630/Topic16/t16_circumference.html



http://moziru.com/explore/Drawn%20triangle%20sphere/

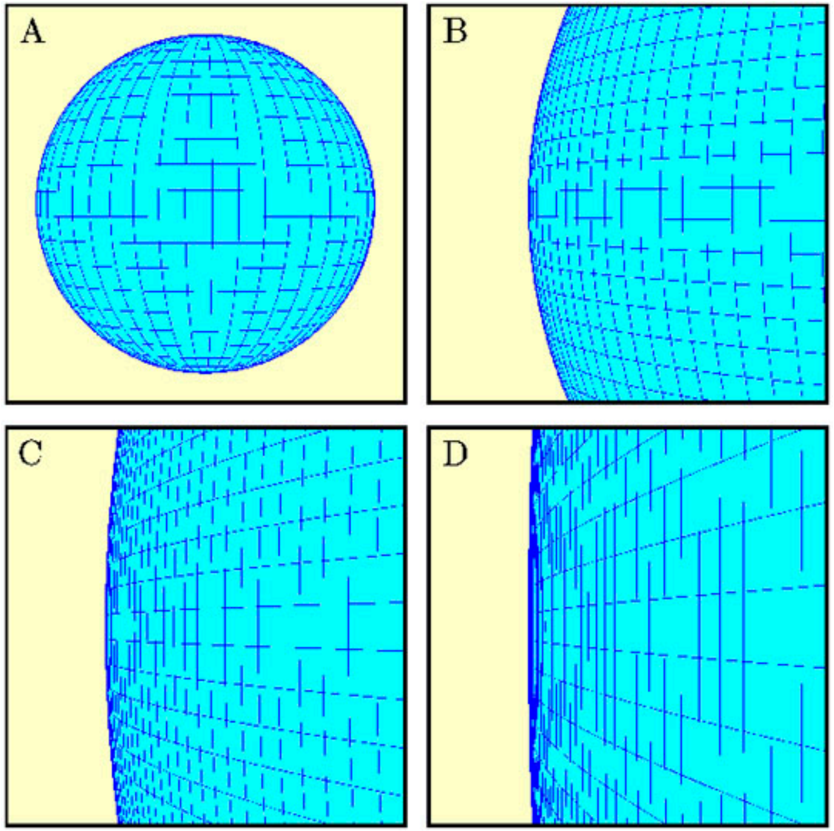
Properties of curved spaces

- It’s easy to visualize curvature by thinking of a *2D curved surface embedded in a 3D Euclidean space*
- However, **curvature is intrinsic to a surface** and can be determined without external reference – Earth dwellers can know the Earth is curved without seeing it from space!



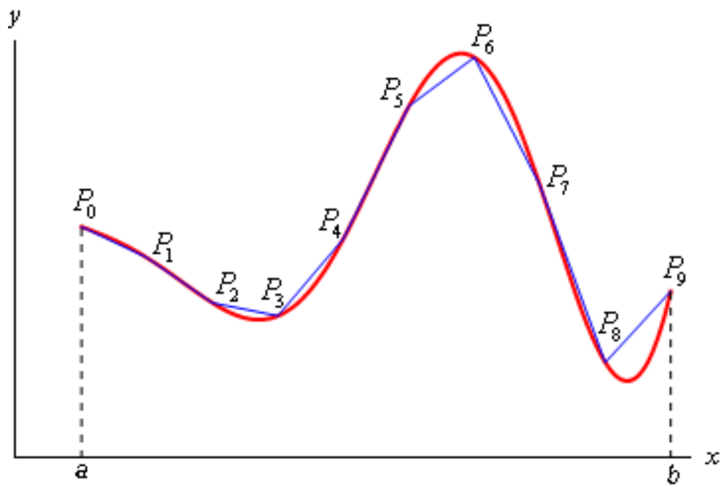
Properties of curved spaces

- Zooming into a small region, **a curved surface is locally flat** (just as a page of an atlas represents a piece of the globe)



https://ned.ipac.caltech.edu/level5/March05/Guth/Guth1.html

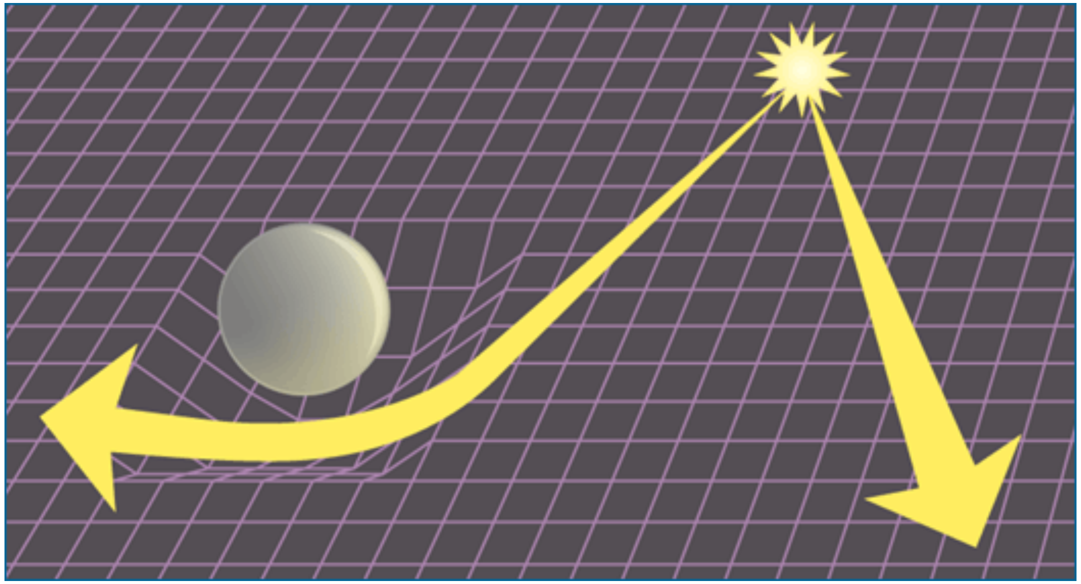
It's analogous to calculus, where we build curves out of straight lines …



http://tutorial.math.lamar.edu/Classes/CalcII/ArcLength.aspx

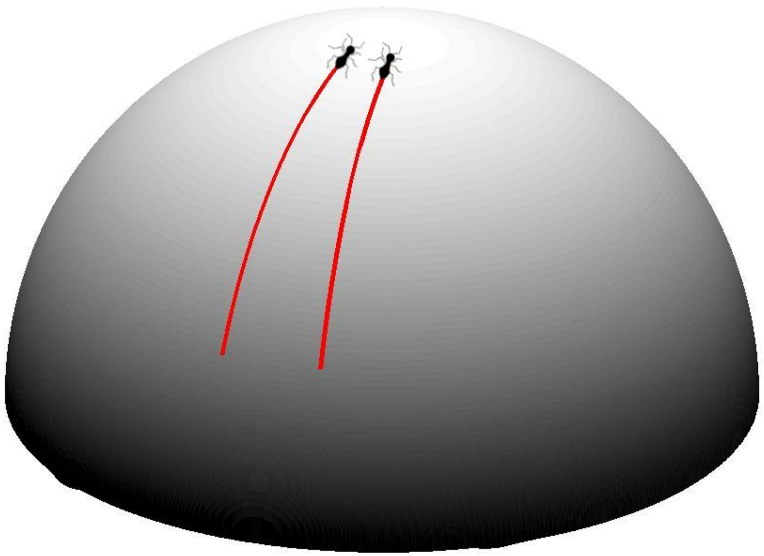
Curvature and gravity

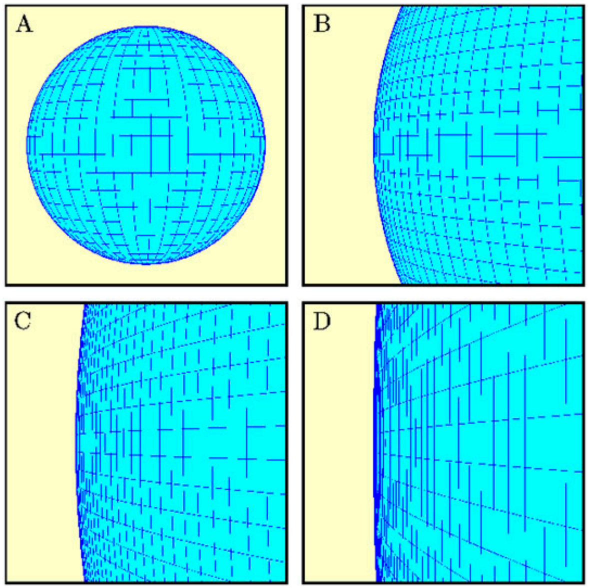
- *What has this got to do with gravity?*
- Gravity can be represented as the **curvature of space-time**
- **Objects travel on a geodesic** in the curved space-time, that *extremizes the space-time interval* between the two points



Curvature and gravity

- Other forces **cause particles to deviate from geodesics** – e.g., *if 2 ants on a curved surface are connected by a solid bar, the force would push them off their geodesics*

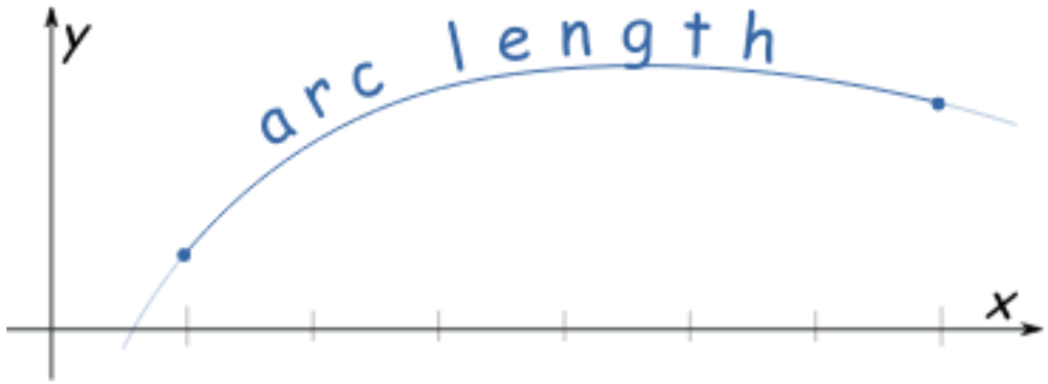




- Just as a curved surface is locally flat, **a curved space-time can locally be described by an inertial frame** (of a freely-falling observer), but there is no extended inertial frame

The metric of a space

- **How do we measure lengths and angles in a Cartesian space with co-ordinates (x, y)?**



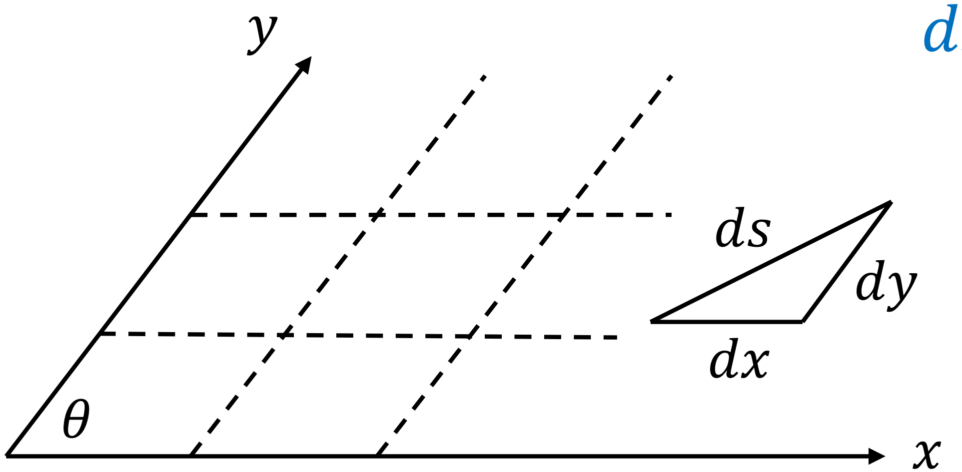
- Breaking the arc into small pieces: $ds^2 = dx^2 + dy^2$

$$s = \int_A^B \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\ d\lambda$$

λ parameterizes the curve [i.e., $x(\lambda)$, $y(\lambda)$]

The metric of a space

- What happens in a *tilted co-ordinate system?*



$$ds^2 = dx^2 + dy^2 - 2\,dx\,dy\,\cos\theta$$
$$= \sum_i \sum_j g_{ij}\,dx^i\,dx^j$$

where $i = \{1,2\}$, $x^{(1)} = x$, $x^{(2)} = y$

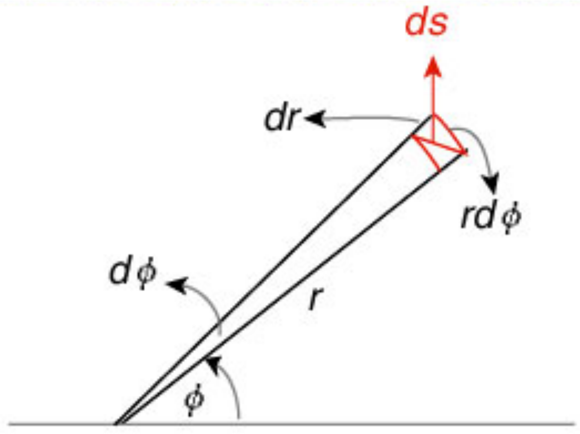
- Using the summation convention, this can be written as $\boldsymbol{ds^2 = g_{ij}\,dx^i\,dx^j}$ where $\boldsymbol{g_{ij}}$ **is the metric of the space**

- In this case, $g_{11} = 1$, $g_{22} = 1$, $g_{12} = g_{21} = -\cos\theta$

The metric of a space

- This example shows that ***the metric determines the geometry, but the geometry does not determine the metric***

- For a given geometry, we can generate many possible metrics through co-ordinate transformations



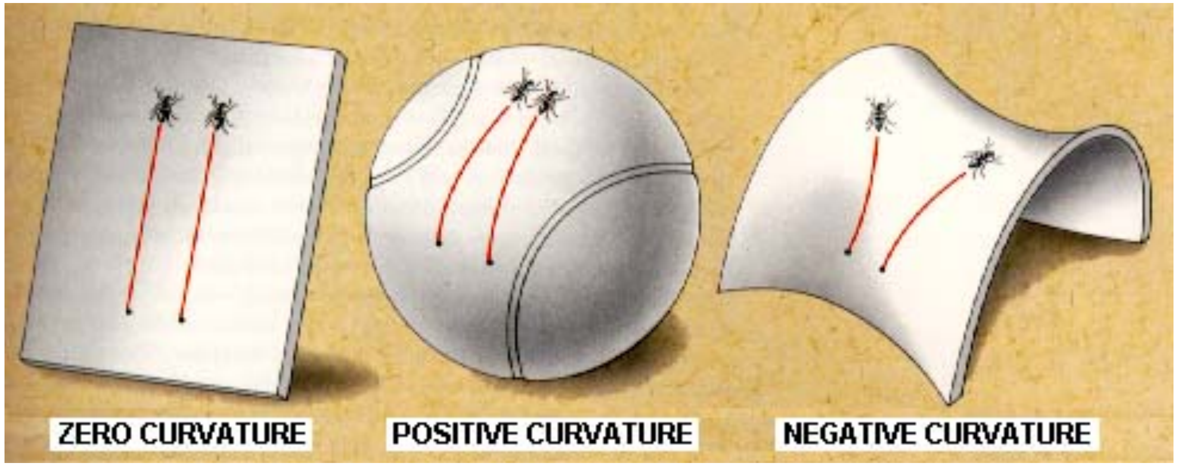
$$(ds)^2 = (dx)^2 + (dy)^2 = (dr)^2 + r^2(d\phi)^2$$

For example – Cartesian and polar co-ordinates

- **The metric encodes information about both the geometry and the co-ordinate system**

How do we define curvature?

- How can observers on a surface **quantify the amount of curvature at a point?**



https://starchild.gsfc.nasa.gov/docs/StarChild/questions/question35.html

- Move a geodesic distance ε in all directions to form a "circle" in the space and measure its area A
- The curvature at a point is then $\lim_{\varepsilon \to 0} \left[\frac{12}{\varepsilon^2} \left(1 - \frac{A}{\pi \varepsilon^2} \right) \right]$

The space-time metric

- The metric of space-time tell us **how to measure the space-time interval between events** (i.e., proper times/distances)

- *Special Relativity*: $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Note that *this geometry is not Euclidean* because $\eta_{00} = -1$ (the locus of events separated by constant ds is a hyperbola, not a circle) – it is known as a **"Minkowski geometry"**

- In *General Relativity*, in a frame containing a gravitational field with curved space-time, $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ where $g_{\mu\nu}$ is the **space-time metric** – a more complicated function of x^μ

The space-time metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

- What is the structure of this function $g_{\mu\nu}$? It's a matrix, where each element is a function of the co-ordinates x^μ

- μ and ν run over 4 indices but, from the above equation, the metric must be symmetric ($g_{\mu\nu} = g_{\nu\mu}$), so there are **10 functions in general** – *this is why GR is complicated!!*

The space-time metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

- Consider a clock at rest in the Earth's frame, which ticks every dt seconds. Is this a proper time interval $d\tau$?
- **No**, because the clock is not in an inertial frame (it is not freely falling)
- The space-time interval between the ticks is $ds^2 = g_{00}(x^i)\,(c\,dt)^2$, since $dx^i = 0$
- Since $ds^2 = -c^2 d\tau^2$, the proper time interval is $\boldsymbol{d\tau = \sqrt{-g_{00}(x^i)}\,dt}$
- **Time runs differently at each point of a gravitational field**

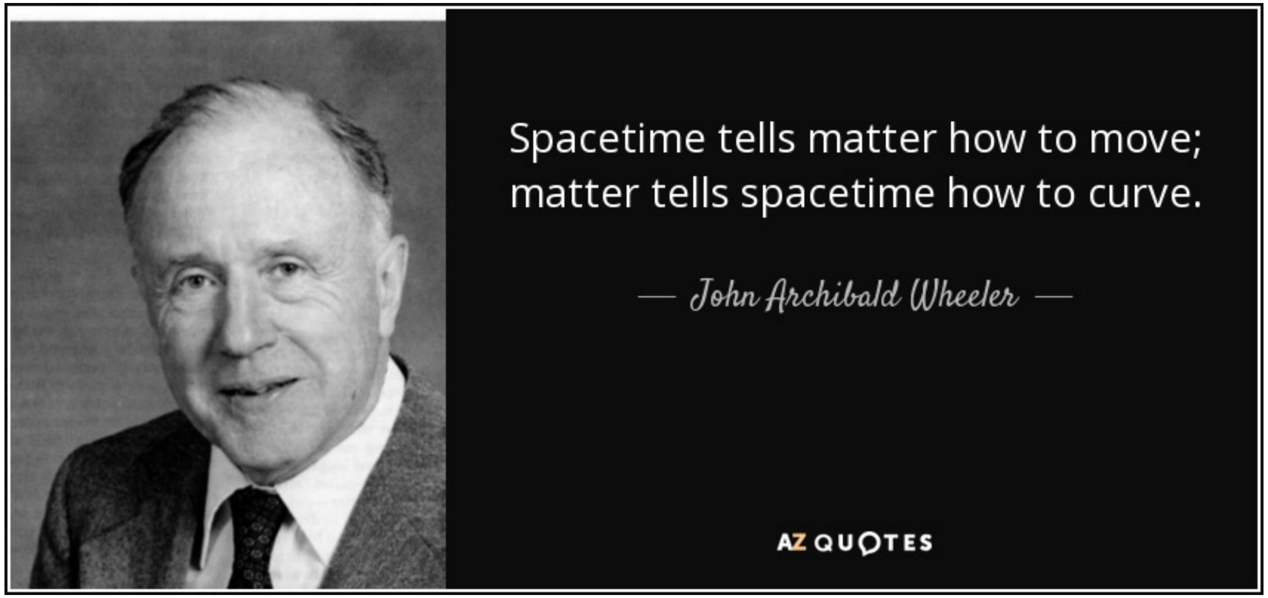
The space-time metric

- A concrete example is a weak gravitational field with potential $\phi(\vec{x})$
- We will show later in the course that: $g_{00}(\vec{x}) = -1 - 2\phi/c^2$
- At height h in a simple vertical gravitational field, $\phi = gh$
- So the ticking period of the clock varies as $dt = d\tau/\sqrt{1 + 2gh/c^2}$

$$d\tau = \sqrt{-g_{00}(\vec{x})}\, dt$$

What determines the metric?

- The question *"what determines the space-time metric"* is the same question as – *"what generates gravity".* The answer is **the distribution of matter and energy**



- Later in the course, we will study the equation which links the space-time curvature to the distribution of matter