19 January 2010 companion to arXiv:0812.3461

van Straten, W. (2009)
The Statistics of Radio Astronomical Polarimetry: Bright Sources and High Time Resolution
The Astrophysical Journal, **694**, 1413.

A Marginal Distributions

The marginal distributions of the Stokes parameters are computed in the natural basis, where the axis of symmetry in cylindrical or spherical coordinates is aligned with the major polarization, S1.

A.1 Equation (19): Instantaneous Intensity

To derive the marginal distribution of the instantaneous total intensity, convert to spherical coordinates using the Jacobian determinant,

$$\left| \frac{\partial (s_1, s_2, s_3)}{\partial (r, \theta, \phi)} \right| = r^2 \cos\phi \tag{1}$$

to arrive at the joint density,

$$f(r,\theta,\phi) = \frac{r\cos\phi}{\pi S^2} \exp\left[-2\left(S_0 r - |\boldsymbol{S}| r \cos\phi \cos\theta\right)/S^2\right]. \tag{2}$$

Note that $r = s_0$ and integrate over θ and ϕ (see `Equation19.nb`) to yield

$$f(s_0) = \frac{2}{|\boldsymbol{S}|} \exp\left(-2\frac{S_0}{S^2} s_0\right) \sinh\left(-2\frac{|\boldsymbol{S}|}{S^2} s_0\right). \tag{3}$$

In terms of the eigenvalues,

$$f(s_0) = (\lambda_0 - \lambda_1)^{-1} \left[\exp\left(-\lambda_0^{-1} s_0\right) - \exp\left(-\lambda_1^{-1} s_0\right)\right]. \tag{4}$$

This distribution has mean S_0 and variance $\|S\|^2/2$.

A.2 Equation (20): Instantaneous Major Polarization

To derive the marginal distribution of the major or natural polarization, convert to cylindrical coordinates using the Jacobian determinant,

$$\left| \frac{\partial\left(s_1, s_2, s_3\right)}{\partial\left(t, \theta, s_1\right)} \right| = t \tag{5}$$

to arrive at the joint density,

$$f(t, \theta, s_1) = \frac{t}{\pi S^2 s_0} \exp\left[-2\left(S_0 s_0 - |\boldsymbol{S}| s_1\right)/S^2\right]. \tag{6}$$

Note that $s_0^2 = t^2 + s_1^2$ and integrate over θ and t (see `Equation20.nb`) to yield

$$f(s_1) = \frac{1}{S_0} \exp\left(-2\frac{S_0|s_1| - |\boldsymbol{S}|s_1}{S^2}\right). \tag{7}$$

In terms of the eigenvalues,

$$f(s_1) = (\lambda_0 + \lambda_1)^{-1} \begin{cases} \exp\left(-\lambda_0^{-1} s_1\right) & s_1 > 0 \\ \exp\left(\lambda_1^{-1} s_1\right) & s_1 < 0 \end{cases}$$

This distribution has mean S_1 and variance $\|S\|^2/2$.

A.3 Equation (21): Instantaneous Minor Polarization

This equation is derived in Appendix A of the paper. The equations in Appendix A are derived in `AppendixA.nb`.

Note that this distribution has mean 0 and variance $S^2/2$.

A.4 Sample Means

The distribution of the sample mean degree of polarization, p, as a function of the number of samples, n, and the populations mean degree of polarizaition, P, are derived in Appendix B of the paper. The equations in Appendix B are derived in `AppendixB.nb`.

B Equation (28): Stokes Covariance Matrix

B.1 The short way

For the simplest derivation of the covariance matrix, note the following:

- A unitary transformation does not alter the degree of polarization; therefore, $\mathbf{b}$ must be Hermitian and $\mathbf{b}^2 = 2\boldsymbol{\rho}$.
- The Mueller matrix $\mathbf{B}$ of the Jones matrix $\mathbf{b}$ is a Lorentz transformation, which is symmetric; therefore $\mathbf{C} = \mathbf{B}^2$.
- The Mueller matrix $\mathbf{B}^2$ corresponds to the Jones matrix $2\boldsymbol{\rho}$, which is also Hermitian; therefore, $\mathbf{C}$ is also a Lorentz transformation.

Refer to Equation (12) of Britton (2000) for the axis-angle parameterization of a Lorentz transformation, and substitute the Jones matrix

$$\mathbf{B}_{\hat{\boldsymbol{m}}}(\beta) = 2\boldsymbol{\rho} = S_i \boldsymbol{\sigma}_i.$$

Here, it is useful to note that

$$\begin{aligned}
\cosh 2\beta &= \quad 2\cosh^2\beta - 1 &&= 2S_0^2 - 1 \\
\sinh 2\beta \, m_k &= \; 2\cosh\beta \sinh\beta \, m_k &&= 2S_0 S_k
\end{aligned}$$

B.2 The long way

Alternatively, you can start with the definition of the covariance matrix, $\mathbf{C} = \mathbf{B}\,\mathbf{B}^{\mathrm{T}}$, or

$$C_{ij} = B_i^k B_j^k = \frac{1}{4} \operatorname{Tr}\left(\boldsymbol{\sigma}_i \, \mathbf{b} \, \boldsymbol{\sigma}_k \, \mathbf{b}^\dagger\right) \operatorname{Tr}\left(\boldsymbol{\sigma}_j \, \mathbf{b} \, \boldsymbol{\sigma}_k \, \mathbf{b}^\dagger\right) \tag{8}$$

and note that the trace is

- a scalar: $a = \operatorname{Tr}(\mathbf{A})$;
- linear: $a \operatorname{Tr}(\mathbf{B}) = \operatorname{Tr}(a\mathbf{B})$;
- commutative: $\operatorname{Tr}(\mathbf{AB}) = \operatorname{Tr}(\mathbf{BA})$; and
- a projection operator: $\mathbf{A} = \operatorname{Tr}(\mathbf{A}\boldsymbol{\sigma}_i)\boldsymbol{\sigma}_i/2$.

Therefore,

$$\begin{aligned}
C_{ij} &= \frac{1}{4}\,\mathrm{Tr}\left[\,\mathrm{Tr}\left(\mathbf{b}^\dagger\,\boldsymbol{\sigma}_i\,\mathbf{b}\,\boldsymbol{\sigma}_k\right)\boldsymbol{\sigma}_k\,\mathbf{b}^\dagger\,\boldsymbol{\sigma}_j\,\mathbf{b}\right] &(9)\\
&= \frac{1}{2}\,\mathrm{Tr}\left(\mathbf{b}^\dagger\,\boldsymbol{\sigma}_i\,\mathbf{b}\,\mathbf{b}^\dagger\,\boldsymbol{\sigma}_j\,\mathbf{b}\right) &(10)\\
&= 2\,\mathrm{Tr}\left(\boldsymbol{\sigma}_i\,\boldsymbol{\rho}\,\boldsymbol{\sigma}_j\,\boldsymbol{\rho}\right), &(11)
\end{aligned}$$

which is the Mueller matrix of $2\boldsymbol{\rho}$. Again, it is possible to refer to Equation (12) of Britton (2000) or use the trace of the anticommutator,

$$\mathrm{Tr}(\{\mathbf{A},\mathbf{B}\}) = 2\,\mathrm{Tr}(\mathbf{AB}) \tag{12}$$

to show that

$$C_{ij} = 2\,\mathrm{Tr}\left(\{\boldsymbol{\sigma}_i\,\boldsymbol{\rho},\boldsymbol{\sigma}_j\,\boldsymbol{\rho}\}\right) = \frac{1}{2}S_k S_l\,\mathrm{Tr}\left(\{\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_k,\boldsymbol{\sigma}_j\,\boldsymbol{\sigma}_l\}\right). \tag{13}$$

The anticommutator of the Pauli matrices,

$$\{\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_j\} = 2\delta_{ij}\boldsymbol{\sigma}_0. \tag{14}$$

Therefore, only 4 of the 16 terms in the above double sum do not vanish.

B.2.1 The Diagonal Elements

When $i = j$, the four $k = l$ terms remain. If $i = j = 0$, the result is simply twice the Frobenius norm,

$$C_{00} = 2\,\mathrm{Tr}(\boldsymbol{\rho}^\dagger\boldsymbol{\rho}) = S_k S_k = 2S_0^2 - |S| \tag{15}$$

For $i = j > 0$, two terms are positive and two are negative:

1. $k = l = 0$: $\mathrm{Tr}(\{\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_i\}) = 4$;
2. $k = l = i = j$: $\mathrm{Tr}(\{\boldsymbol{\sigma}_0,\boldsymbol{\sigma}_0\}) = 4$;

3–4. otherwise: $\mathrm{Tr}(\{i\epsilon_{ik\alpha}\boldsymbol{\sigma}_\alpha, i\epsilon_{ik\alpha}\boldsymbol{\sigma}_\alpha\}) = 4i^2 = -4$.

Therefore,

$$C_{\alpha\alpha} = S_0^2 + S_\alpha^2 - S_\beta^2 - S_\gamma^2 = 2S_\alpha^2 + |S|, \tag{16}$$

where α, β, and γ are all greater than zero and unequal. Combining the two equations yields

$$C_{ii} = 2S_i^2 - \eta_{ii}|S| \tag{17}$$

(no summation).

B.2.2 The Off-Diagonal Elements

When $i \neq j$, the four terms that remain are

1. $k = i$ and $l = j$: the arguments to the anticommutator are the identity matrix and the trace is 4;

2. $k = j$ and $l = i$: if either i or j is zero, then the arguments to the anticommutator are the other matrix and the trace is 4; otherwise, $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = -\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = i\epsilon_{ijk}\boldsymbol{\sigma}_k$ and the trace is $-4i^2 = 4$.

3. $k = 0$ and $\epsilon_{jli} \neq 0$: $\mathrm{Tr}(\{\boldsymbol{\sigma}_i, \pm i\boldsymbol{\sigma}_i\}) = \pm 4i$;

4. $l = 0$ and $\epsilon_{ikj} \neq 0$: $\mathrm{Tr}(\{\mp i\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_j\}) = \mp 4i$.

The last two terms cancel eachother, and $C_{ij} = 2S_iS_j$.

B.3 In the Eigen Basis

In the eigen basis, the covariance matrix

$$\mathbf{C} = \begin{pmatrix} S_0^2 + |\boldsymbol{S}|^2 & 2S_0|\boldsymbol{S}| & 0 & 0 \\ 2S_0|\boldsymbol{S}| & S_0^2 + |\boldsymbol{S}|^2 & 0 & 0 \\ 0 & 0 & S^2 & 0 \\ 0 & 0 & 0 & S^2 \end{pmatrix} \tag{18}$$

from which it is trivial to derive the determinant by Laplacian expansion,

$$|\mathbf{C}| = \left(S_0^2 + |\boldsymbol{S}|^2\right)^2 S^4 - \left(2S_0|\boldsymbol{S}|\right)^2 S^4 = S^8 \tag{19}$$