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van Straten, W. (2006)
Radio Astronomical Polarimetry and High-Precision Pulsar Timing
The Astrophysical Journal, **642**, 1004.

A Equation (13): Conditional Variance of φ

Equation (13) presents an analytical expression for the variance of the best-fit phase shift. Given only the template profile, it can be used to predict the arrival time precision as a function of the S/N of the observation.

A.1 Derivation

Start by deriving Equation (11), noting that Equation (10) can be written

$$\chi^2 = \sum_{m=1}^{N/2} \sum_{k=0}^{3} \varsigma_k^{-2} \Delta_{k,m} \Delta_{k,m}^*, \tag{1}$$

where

$$\Delta_{k,m} = S_k'(\nu_m) - \mathrm{tr}[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'(\nu_m)], \tag{2}$$

such that

$$\frac{\partial \chi^2}{\partial \varphi} = \sum_{m=1}^{N/2} \sum_{k=0}^{3} \varsigma_k^{-2} 2\mathrm{Re}\left[\Delta_{k,m}^* \frac{\partial \Delta_{k,m}}{\partial \varphi} \right]. \tag{3}$$

Now

$$\frac{\partial \Delta_{k,m}}{\partial \varphi} = i2\pi\nu_m \mathrm{tr}\left[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'(\nu_m)\right] \tag{4}$$

and $\mathrm{Re}[iz] = -\mathrm{Im}[z]$; therefore,

$$\frac{\partial \chi^2}{\partial \varphi} = -4\pi \sum_{m=1}^{N/2} \sum_{k=0}^{3} \varsigma_k^{-2} \nu_m \mathrm{Im}\left[\Delta_{k,m}^* \mathrm{tr}\left[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'(\nu_m)\right] \right]. \tag{5}$$

In the above equation, the second term of $\Delta_{k,m}^*$ is multiplied by its complex conjugate, resulting in a real number (the squared modulus) with no imaginary component; therefore,

$$\frac{\partial \chi^2}{\partial \varphi} = -4\pi \sum_{m=1}^{N/2} \sum_{k=0}^{3} \varsigma_k^{-2} \nu_m \mathrm{Im}\left[S_k'^*(\nu_m) \mathrm{tr}\left[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'(\nu_m)\right] \right]. \tag{6}$$

Now define the cross-spectral power of the template and the observation

$$S_{k,m} = S'^{*}_{k}(\nu_m)\mathrm{tr}\left[\boldsymbol{\sigma}_k\ \boldsymbol{\rho}_0(\nu_m)\right], \tag{7}$$

such that

$$\frac{\partial\chi^2}{\partial\varphi} = -4\pi\sum_{m=1}^{N/2}\sum_{k=0}^{3}\varsigma_k^{-2}\nu_m\mathrm{Im}\left[S_{k,m}\exp(-i2\pi\nu_m\varphi)\right]. \tag{8}$$

Equation (11) is obtained after expressing $S_{k,m}$ in polar coordinates; Equation (12) is a small angle (or high S/N) approximation. Note that the opening minus sign is missing in these equations.

To derive Equation (13), first solve Equation (12) for φ

$$\varphi = \frac{\sum|S_{k,m}|\phi_{k,m}\nu_m}{2\pi\sum|S_{k,m}|\nu_m^2}, \tag{9}$$

where $\sum = \sum_{m=1}^{N/2}\sum_{k=0}^{3}$ is introduced for convenience. Standard error propagation is greatly simplified by recognizing that the variance of $\phi_{k,m}$ is $|S_{k,m}|$. Therefore, to first order, the variance of φ is given by

$$\mathrm{var}(\varphi) = \sum_{m=1}^{N/2}\sum_{k=0}^{3}\left(\frac{\partial\varphi}{\partial\phi_{k,m}}\right)^2|S_{k,m}| \tag{10}$$

Substitution of

$$\frac{\partial\varphi}{\partial\phi_{k,m}} = \frac{|S_{k,m}|\nu_m}{2\pi\sum|S_{k,m}|\nu_m^2} \tag{11}$$

into equation 10 yields

$$\mathrm{var}(\varphi) = \sum_{m=1}^{N/2}\sum_{k=0}^{3}\frac{|S_{k,m}|\nu_m^2}{4\pi^2\left(\sum|S_{k,m}|\nu_m^2\right)^2} = \frac{1}{4\pi^2\sum|S_{k,m}|\nu_m^2} \tag{12}$$

and after expanding $\sum$

$$\mathrm{var}(\varphi) = \left[4\pi^2\sum_{m=1}^{N/2}\nu_m^2\sum_{k=0}^{3}|S_{k,m}|\varsigma_k^{-2}\right]^{-1}. \tag{13}$$

A.2 Interpretation

1. Phases such as $\phi_n = n/N$ and φ are dimensionless turns; therefore, TOA $\tau = \tau_0 + \varphi P$, where P is the pulse period.

2. Frequencies such as $\nu_m = m$ are dimensionless harmonics.

3. $S_{k,m}$ is the cross spectral power between the template and profile; in the case of the theoretical prediction, it is the autospectral power in the template.

4. The r.m.s. of the noise in the Fourier domain ς_k is a function of the S/N of the observation.

5. Equation (13) returns variance in dimensionless phase, which can be translated into arrival time error with $\sigma_\tau = \sqrt{\mathrm{var}(\varphi)}P$.

B Equation (14): Curvature Matrix

Begin with the definition of the merit function,

$$\chi^2 = \sum_{m=1}^{N/2} \sum_{k=0}^{3} |S'_k(\nu_m) - \mathrm{tr}[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'(\nu_m)]|^2 \varsigma_k^{-2}. \tag{14}$$

Letting $\boldsymbol{\rho}'_m = \boldsymbol{\rho}'(\nu_m)$ and $\varsigma_k = \varsigma$, the first partial derivative is

$$\frac{\partial \chi^2}{\partial \eta_r} = -\frac{2}{\varsigma^2} \sum_{m=1}^{N/2} \sum_{k=0}^{3} \mathrm{Re} \left[(S'_k(\nu_m) - \mathrm{tr}[\boldsymbol{\sigma}_k \, \boldsymbol{\rho}'_m]) \mathrm{tr} \left(\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_r} \right)^* \right] \tag{15}$$

Because the Pauli spin matrices are Hermitian, $\mathrm{tr}(\boldsymbol{\sigma}_k \, \boldsymbol{\rho})^* = \mathrm{tr}(\boldsymbol{\sigma}_k \, \boldsymbol{\rho}^\dagger)$. The second partial derivative is

$$\frac{\partial^2 \chi^2}{\partial \eta_r \partial \eta_s} = \frac{2}{\varsigma^2} \sum_{m=1}^{N/2} \sum_{k=0}^{3} \mathrm{Re} \left[\mathrm{tr} \left(\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s} \right) \mathrm{tr} \left(\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'^\dagger_m}{\partial \eta_r} \right) \right]. \tag{16}$$

Following the discussion in § 15.5 of Numerical Recipes, the term containing a second derivative in equation (20) has been dropped. Now the trace of a matrix is a scalar and $c\,\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(c\mathbf{A})$; therefore,

$$\frac{\partial^2 \chi^2}{\partial \eta_r \partial \eta_s} = \frac{2}{\varsigma^2} \sum_{m=1}^{N/2} \sum_{k=0}^{3} \mathrm{Re} \left[\mathrm{tr} \left(\mathrm{tr} \left[\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s} \right] \boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'^\dagger_m}{\partial \eta_r} \right) \right] \tag{17}$$

Furthermore $\mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B}) = \mathrm{tr}(\mathbf{A} + \mathbf{B})$ and $\mathrm{Re}(z) + \mathrm{Re}(w) = \mathrm{Re}(z + w)$, so

$$\frac{\partial^2 \chi^2}{\partial \eta_r \partial \eta_s} = \frac{2}{\varsigma^2} \sum_{m=1}^{N/2} \mathrm{Re}\left[\mathrm{tr}\left(\left[\sum_{k=0}^{3} \mathrm{tr}\left(\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s}\right) \boldsymbol{\sigma}_k\right] \frac{\partial \boldsymbol{\rho}'^{\dagger}_m}{\partial \eta_r}\right)\right]. \tag{18}$$

The first factor inside the trace,

$$\sum_{k=0}^{3} \mathrm{tr}\left(\boldsymbol{\sigma}_k \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s}\right) \boldsymbol{\sigma}_k = 2\frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s} \tag{19}$$

and $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$; therefore

$$\frac{\partial^2 \chi^2}{\partial \eta_r \partial \eta_s} = \frac{4}{\varsigma^2} \sum_{m=1}^{N/2} \mathrm{Re}\left[\mathrm{tr}\left(\frac{\partial \boldsymbol{\rho}'^{\dagger}_m}{\partial \eta_r} \frac{\partial \boldsymbol{\rho}'_m}{\partial \eta_s}\right)\right]. \tag{20}$$

C Equation (28): Non-orthogonality, $\sin(\delta)$

Begin with

$$\cos 2\delta = -\cos\Theta = -\frac{\boldsymbol{S}_0 \cdot \boldsymbol{S}_1}{|\boldsymbol{S}_0||\boldsymbol{S}_1|} = 1 - 2\left(\frac{|\boldsymbol{r}_0^{\dagger}\boldsymbol{r}_1|}{|\boldsymbol{r}_0||\boldsymbol{r}_1|}\right)^2 \tag{21}$$

then use the double-angle formula $\cos 2\delta = 1 - 2\sin^2\delta$.

D Addendum: Systematic Timing Error

It is possible to predict the systematic timing error due to polarization calibration errors by computing the rate at which the best estimate of the phase shift varies with respect to variations in the instrumental boost transformation parameters. Beginning with Equation (9) of this companion, the best estimate of the phase shift derived from the total intensity alone is given by

$$\varphi = \frac{\sum \phi_{0,m}|S_{0,m}|\nu_m}{2\pi \sum |S_{0,m}|\nu_m^2}, \tag{22}$$

where the summation is performed over $m = 1$ to $m = N/2$. Now consider an observation that is a copy of the template subjected to an instrumental boost transformation, as parameterized in Section 4.1 of van Straten (2004, ApJSS 152:129) such that $\sinh^2\beta = \boldsymbol{b}\cdot\boldsymbol{b}$. Referring to Equation (10) of Britton (2000, ApJ 532:1240), the transformed total intensity is given by

$$S_0' = (1 + 2\,\boldsymbol{b}\cdot\boldsymbol{b})S_0 + 2b_0\,\boldsymbol{b}\cdot\boldsymbol{S}, \tag{23}$$

where $b_0 = \cosh\beta = \sqrt{1+\boldsymbol{b}\cdot\boldsymbol{b}}$. The partial derivatives of S_0' with respect to the boost parameters b_k are simply

$$\frac{\partial S_0'}{\partial b_k} = 4b_kS_0 + 2\frac{b_k}{b_0}\,\boldsymbol{b}\cdot\boldsymbol{S} + 2b_0S_k; \tag{24}$$

therefore

$$\left.\frac{\partial S_{0,m}}{\partial b_k}\right|_{\beta=0} = 2S_k^*(\nu_m)S_0(\nu_m). \tag{25}$$

Furthermore, if $S_{0,m} = X_m + iY_m$ and

$$\phi_{0,m} = \tan^{-1}\frac{Y_m}{X_m}. \tag{26}$$

then

$$\frac{\partial \phi_{0,m}}{\partial b_k} = \frac{1}{|S_{0,m}|^2}\left(X_m\frac{\partial Y_m}{\partial b_k} - Y_m\frac{\partial X_m}{\partial b_k}\right). \tag{27}$$

When $\beta = 0$, $Y_m = 0$ and $X_m = S_{0,m}$, the partial derivatives of the phase shift with respect to the boost parameters reduces to

$$\dot\varphi_k = \left.\frac{\partial\varphi}{\partial b_k}\right|_{\beta=0} = \frac{\sum \mathrm{Im}[S_k^*(\nu_m)S_0(\nu_m)]\nu_m}{\pi\sum|S_0(\nu_m)|^2\nu_m^2}. \tag{28}$$

The above expression for $k = 1,2,3$ defines the gradient vector $\dot{\boldsymbol{\varphi}}$. This gradient may be used to predict the susceptability of arrival time estimates to instrumental boost transformations arising from differential gain and non-orthogonality of the rececptors, defined as the magnitude of the gradient,

$$\dot\varphi_\beta = |\dot{\boldsymbol{\varphi}}|. \tag{29}$$