Order-of-magnitude questions on Planet Formation

Some useful numbers:

$$M_\odot \simeq 2 \times 10^{33}\,\mathrm{gm}$$
$$1\,\mathrm{AU} \simeq 1.5 \times 10^{13}\,\mathrm{cm}$$

And a useful reference:

Lissauer J., "Planet formation", 1993, *Annu. Rev. Astron. Astrophys.* 31:129–174

Protoplanetary Disk

1. Show that the masses of the three groups of planets – terrestrial; giant (Jupiter and Saturn); outer (Neptune and Uranus) – each required $\sim 10^{-2}\,M_\odot$ of cosmic-composition material.

2. Assuming that the disk had mass $10^{-2}\,M_\odot$ within about $1\,\mathrm{AU}$, what was the surface density in (a) gas, (b) silicates?

3. Show that the thickness of the disk $h \sim c_\mathrm{s}/\Omega$, where c_s is the thermal sound speed and Ω is the angular velocity. What was the thickness of the disk at radius $a \sim 1\,\mathrm{AU}$?

4. Using the answers to the last two questions, what was the volume density of gas at radius $a \sim 1\,\mathrm{AU}$?

5. Show that the orbital velocity of the gas is less than the Keplerian velocity by $\Delta v \sim -\frac{c_\mathrm{s}^2}{\Omega a}$. (Hint: Lissauer §2.)

6. Thus show that all particles spiral into the sun (Hint: Lissauer §2). Show that the fastest rate of infall occurs for particles of radius $R \sim 1\,\mathrm{m}$ and that this rate is $\sim 10^6\,\mathrm{km\,yr^{-1}}$.

7. Grain growth. Assume a small refractory grain acts as a nucleation centre. Consider condensation of a species comprising a fraction f of the total gas density and having atomic weight A. How fast can the radius R of the grain grow?

Planetesimal Dynamics

1. Show that the solar tidal field is comparable to the direct gravity of a planetesimal at the boundary of the Hill sphere. (Pages 143 onwards of Lissauer will be very useful in answering the questions in this section.)

2. a) Show that random velocity dominates Keplerian shear for accreted particles if $e_\mathrm{H} \gg 1$. **b)** and in this case prove that planetesimal–planetesimal interaction time$\ll$orbital period.

3. Derive Lissauer's equation 12 for the gravitational enhancement of the collisional cross-section.

4. a) Think about why gravitational scatterings (no physical collisions) tend to increase e and i in a differentially rotating disk. **b)** Note that this would not happen inside a box containing N gravitationally interacting points particles if their random velocity was large enough such that the density was nearly uniform.

Giant Planet Atmospheres

1. This question shows why the critical core size is a few $M_\oplus$. **a)** Consider a core of mass M and radius R in an isothermal gas whose density and sound speed at infinity are ρ_0 and c_s. Show that if the envelope remains isothermal (i.e., it can radiate away gravitional energy in an accretion time), then it will have mass

$$M_{\rm env} \sim 4\pi R^2 \frac{c_s^2 R^2}{GM} \rho_0 \exp\left(\frac{GM}{Rc_s^2}\right).$$

b) Use this to find M such that $M \sim M_{\rm env}$.

Disks around Young Stars

1. a) Optical depth. Consider the disk described in question 2 of the first section. Calculate the optical depth to black-body radiation of the ambient temperature, assuming that all heavy elements are in grains of size ~ 0.1 micron. Assume that the grain opacity scales with wavelength as $\kappa \sim \lambda^{-p}$, $1 < p < 2$, for $\lambda > 2\pi r$. **b)** Suppose that the surface density and temperature of the disk vary with radius like $\Sigma \propto a^{-3/2}$ and $T \propto a^{-3/4}$. At what radius does $\tau \sim 1$?

2. Disk temperature from stellar heating. Assume that the central star has radius R_* and temperature T_*. Derive the temperature of the disk (assumed thin) at radius a from stellar heating.

3. Disk temperature from accretion. **a)** Show that the accretion luminosity is $F \sim \frac{\dot{M}}{2\pi a}\frac{GM_*}{a^2}$. What disk temperature distribution does this lead to? **b)** Can the disk temperature from accretion exceed the disk temperature from stellar luminosity?

4. Disk spectral energy distribution in the optically thick limit. Using the results above, show that $\nu F_\nu \propto \nu^{4/3}$.

5. Suppose that at $a \sim 1\,$AU, the disk's temperature $T \sim 300\,$K and surface density $\Sigma \sim 10^4\,{\rm g\,cm^{-2}}$. Derive an estimate for the cooling time. Note that the cooling time must be much less than the accretion time for the thin disk assumption to be consistent.

6. a) Derive the viscous torque in an accretion disk across semi-major axis a. **b)** Show that the accretion time, $t_{\rm acc} \sim a^2/\nu$. **c)** Evaluate $t_{\rm acc}$ for molecular viscosity at 1 AU in the protoplanetary disk. **d)** Assuming turbulent viscosity with a maximum eddy size $h \sim c_s/\Omega$ and $v_{\rm turb} \sim \mathcal{M} c_s$ (where $\mathcal{M} \lesssim 1$ is the Mach number), estimate $t_{\rm acc}$.