Class 9: Black Holes

In this class we will study an exact solution of General Relativity known as the Schwarzschild Metric, which describes the unusual space-time properties around a Black Hole

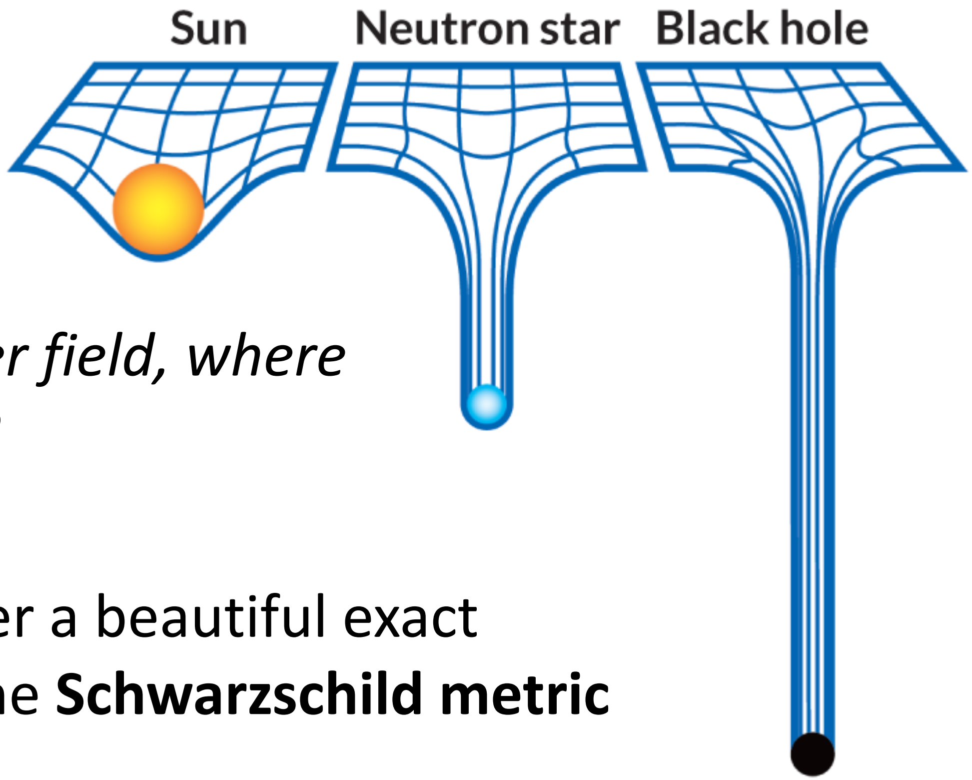
Class 9: Black Holes

At the end of this session you should be able to …

- … describe the space-time around an object using the **Schwarzschild metric**
- … understand the effect of the metric on **clock rates** and **gravitational redshifting** of light emitted near the object
- … solve for the **motion of light rays and freely-falling objects** on radial and circular paths near a black hole
- … describe effects taking place at the **Schwarzschild Radius**

Strong-field gravity

- We have seen how General Relativity recovers Newtonian gravity in the case of a “weak field”, such as near the Sun



- *What happens in a stronger field, where Newton’s Laws don’t hold?*

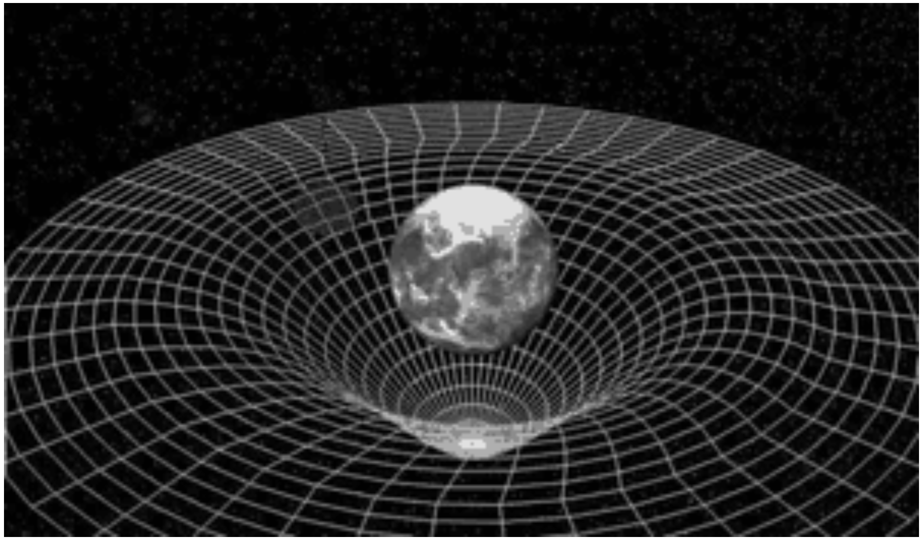
- In this class we will consider a beautiful exact solution of such a case – the **Schwarzschild metric**

http://scienceblogs.com/startswithabang/2015/06/30/astroquizzical-how-does-gravity-escape-from-a-black-hole-synopsis/

Schwarzschild metric

- We know that *space-time curvature is completely specified by the metric* $g_{\mu\nu}$, such that $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$
- Schwarzschild found the metric for the **empty space around a spherically-symmetric, static, matter distribution**

K.Schwarzschild (1873-1916)



https://www.physicsoftheuniverse.com/scientists_schwarzschild.html
http://www.skyandtelescope.com/astronomy-resources/are-black-holes-real/

Schwarzschild metric

- The **Schwarzschild metric** is expressed in terms of space-time co-ordinates (ct, r, θ, ϕ), and is:

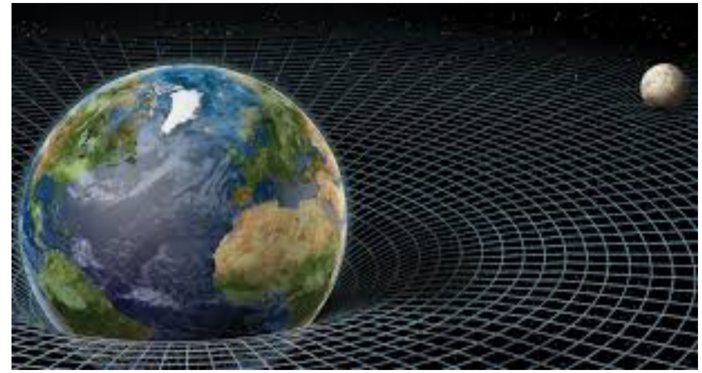
$$\boxed{ds^2 = -\left(1 - \frac{R_s}{r}\right)(c\,dt)^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2[d\theta^2 + (\sin\theta\ \ d\phi)^2]}$$

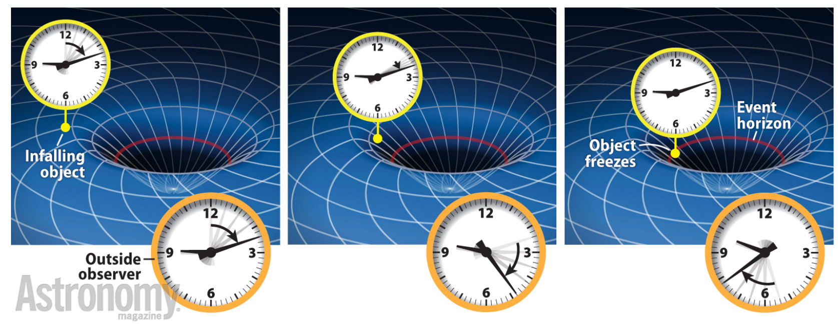
- We can see the non-zero components of the metric $g_{\mu\nu}$ are $g_{tt} = -\left(1 - \frac{R_s}{r}\right)$, $g_{rr} = \left(1 - \frac{R_s}{r}\right)^{-1}$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = (r\sin\theta)^2$

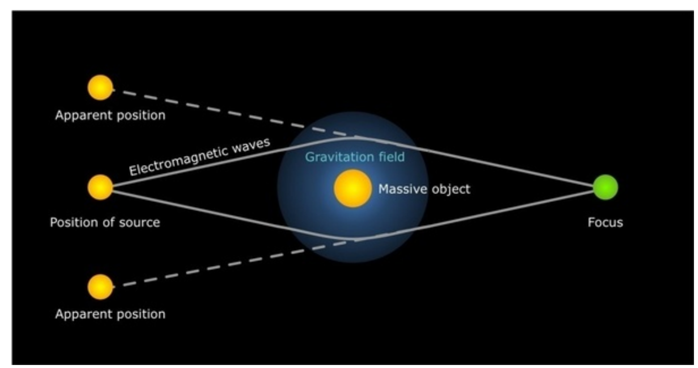
- $R_s = 2GM/c^2$ is the **Schwarzschild radius** in terms of the total mass enclosed, M

Schwarzschild metric

- This metric tells us how an object curves the space-time around it, and can be used to compute …
- … the **orbits of planets** (which are in free-fall around the central object)
- … the **gravitational time dilation** around black holes
- … the **deflection of light** by a gravitational field



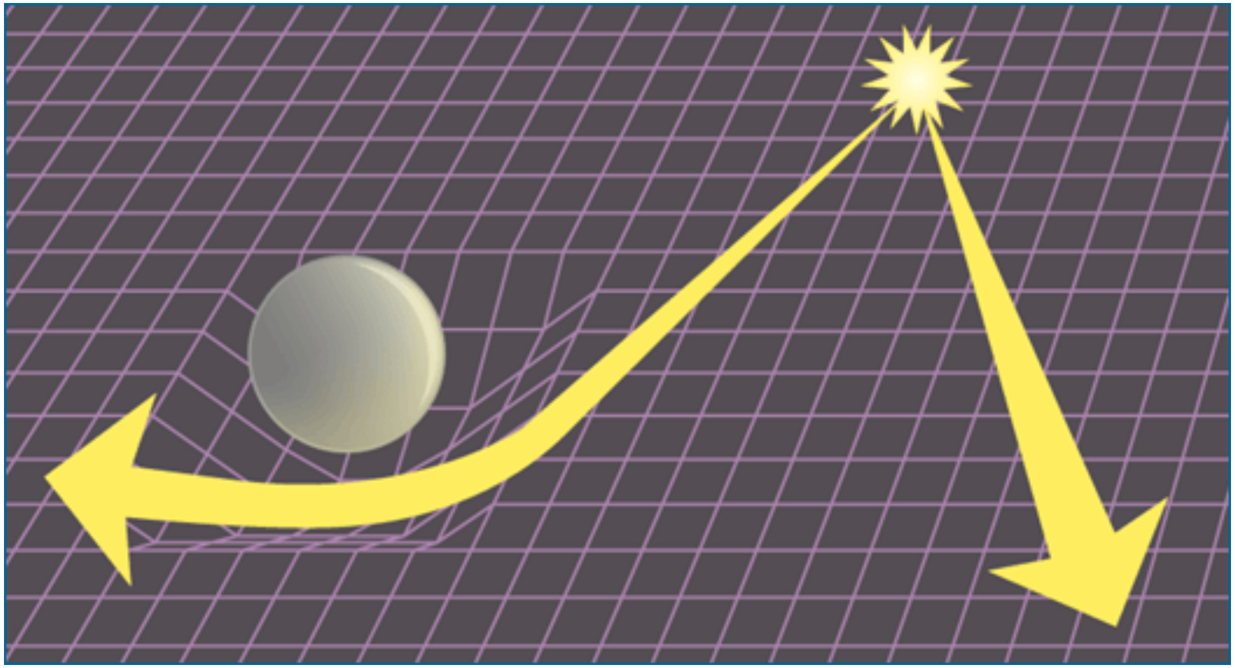




http://www.astronomy.com/magazine/ask-astro/2014/11/frozen-stars
https://www.quora.com/What-is-gravitational-deflection-of-light-rays
https://www.sciencenews.org/article/einsteins-genius-changed-sciences-perception-gravity

Schwarzschild metric

- Note that the space-time around the black hole is **empty**, but still **curved by the nearby mass** (e.g., particles will move on curved orbits as seen by a distant observer)



Schwarzschild metric

- Let's break down the Schwarzschild metric in more detail, by comparing it to the Minkowski metric:

Minkowski: $$ds^2 = -(c\,dt)^2 + dr^2 + r^2[d\theta^2 + (\sin\theta\ d\phi)^2]$$

Schwarzschild: $$ds^2 = -\left(1 - \frac{R_s}{r}\right)(c\,dt)^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2[d\theta^2 + (\sin\theta\ d\phi)^2]$$

Radial distortions in g_{tt} and g_{rr}

Normal spherical co-ordinates for $g_{\theta\theta}$ and $g_{\phi\phi}$

- Note that r is a radial co-ordinate, but **not a distance** – the **proper distance** measured by a local observer is $dL = \frac{dr}{\sqrt{1-R_s/r}}$

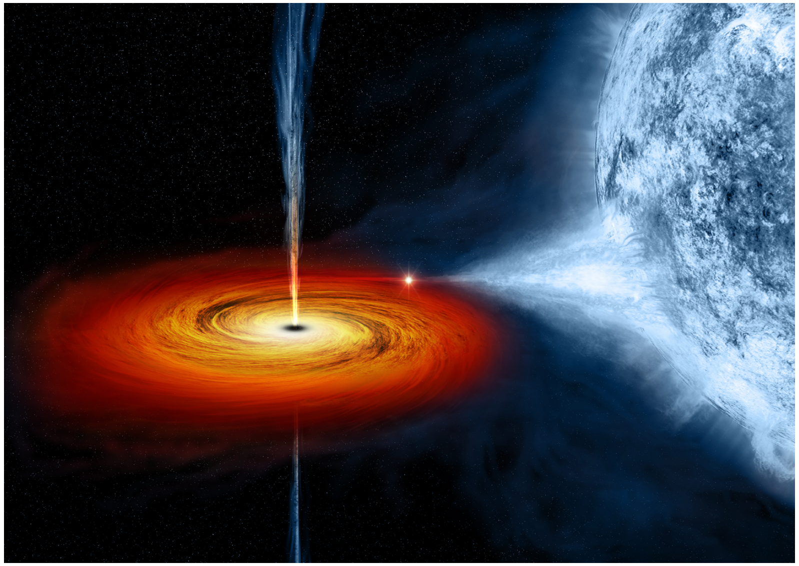
- Space-time co-ordinates are "like street numbers" – that is, *not sufficient to determine distances without the metric!*

Gravitational time dilation

- Consider a ticking clock, fixed in place at different radii r
- Since $dr = d\theta = d\phi = 0$, the **co-ordinate time** dt between two ticks (recorded by the reference frame) is related to the **proper time** $d\tau$ (recorded by the clock) as $dt = \frac{d\tau}{\sqrt{1-\frac{R_S}{r}}}$
- At large r, $dt = d\tau$ (there is no distinction between times)
- As r decreases towards R_s, the co-ordinate time dt between the ticks increases
- At $r = R_s$, $dt = \infty$ and in co-ordinate time, the clock stops!!
- **Something weird is happening at the Schwarzschild radius!**

Black holes

- For almost all objects (e.g. the Earth, Sun), $\boldsymbol{r = R_s}$ **lies inside the surface**, so these effects never apply (N.B. the Schwarzschild metric describes *empty space* outside the object)
- An object whose size is less than its Schwarzschild radius is called a **black hole** – and they exist in astrophysics!!



e.g. (1) a black hole is produced at the end of a star's life, when nuclear fusion is over;

(2) supermassive black holes develop at the centres of galaxies

https://www.nasa.gov/audience/forstudents/k-4/stories/nasa-knows/what-is-a-black-hole-k4.html

The Schwarzschild Radius

- We can gain more intuition for R_s using the result from Class 4 relating the clock rate, $C = d\tau/d\tau_\infty$, to the proper acceleration α that is equivalent to the gravitational field: $\alpha = c^2 \frac{d}{dL} \ln C$
- Here, $C = \sqrt{1 - \frac{R_s}{r}}$ and proper distance $dL = \frac{dr}{\sqrt{1-\frac{R_s}{r}}}$
- We find: $\alpha = -\frac{R_s\, c^2/r^2}{2\sqrt{1-R_s/r}}$
- Far from the black hole, the equivalent proper acceleration is $\alpha = -GM/r^2$, just as Newton would have predicted
- At $r = R_s$, $\alpha = \infty$: **it takes an infinite proper acceleration to remain static, i.e. the clock "feels infinitely heavy"**

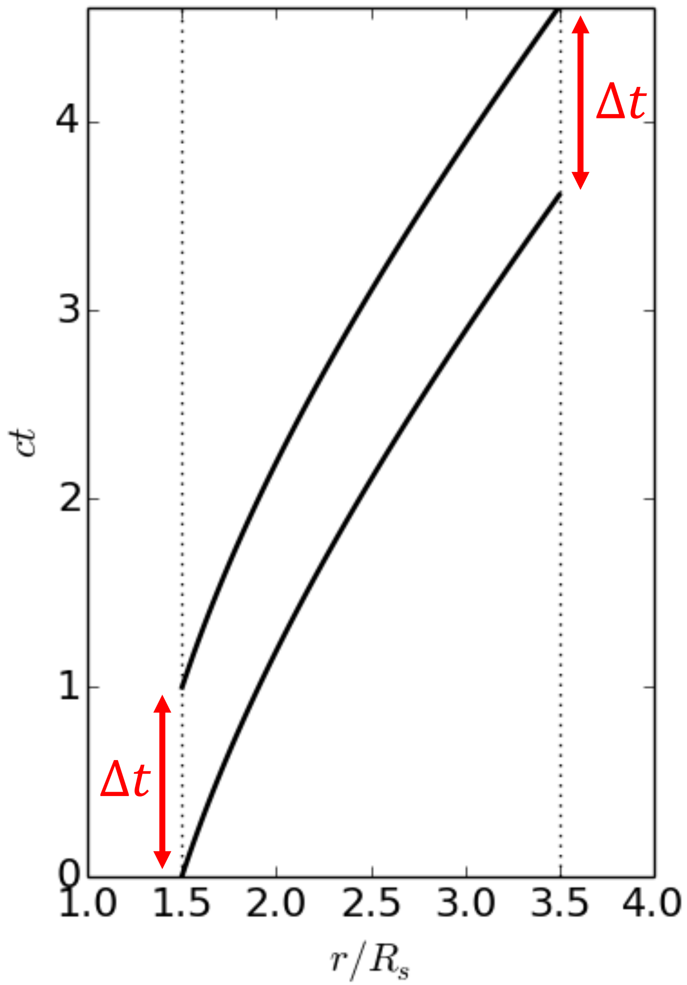
Light rays near a black hole

- *Consider shining a torch radially outwards near a black hole!*

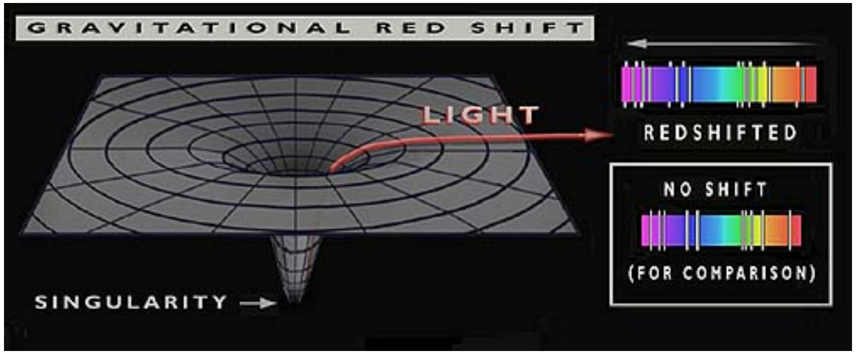
https://warwick.ac.uk/newsandevents/pressreleases/black_hole_kills/

Light rays near a black hole

Consider two successive crests of a radial light ray:



- The co-ordinate time Δt between the crests remains the same, but the **proper time** between the crests varies as $\Delta\tau = \sqrt{1 - \frac{R_s}{r}}\,\Delta t$
- The **frequency** f of the light scales as $1/\Delta\tau$ – light emitted from r will experience a **gravitational redshift** $1 + z = \frac{f_r}{f_\infty} = \left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}}$



http://archive.ncsa.illinois.edu/Cyberia/NumRel/EinsteinTest.html

Light rays near a black hole

- **Let's solve for the world-line of the radial light ray $\boldsymbol{r(t)}$**
- A light ray connects space-time events separated by $ds = 0$
- For a radially-moving light ray, $d\theta = d\phi = 0$
- The metric is then $ds^2 = -\left(1 - \frac{R_s}{r}\right)(c\ dt)^2 + \frac{dr^2}{1-\frac{R_s}{r}}$
- Hence $ds = 0$ implies $\frac{1}{c}\frac{dr}{dt} = 1 - \frac{R_s}{r}$
- We can solve this equation to give $ct = R_s \ln(r - R_s) + r + K$

Light rays near a black hole

- At the Schwarzschild radius $r = R_s$, $\frac{dr}{dt} = 0$ – which implies that **all radial light rays remain at the horizon!**
- A light ray at the horizon directed a little bit sideways must move to smaller r – **there are no paths to escape $\boldsymbol{r < R_s}$**

http://www.wired.co.uk/article/what-black-holes-explained

Radial plunge into a black hole

- We’ll now look at a material object falling into a black hole!

https://www.nature.com/news/astrophysics-fire-in-the-hole-1.12726

Radial plunge into a black hole

- Freely-falling observers with world lines $x^\mu(\tau)$ follow geodesics $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$, where the Christoffel symbols are given by $\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$

- We can use the $\mu = t$ geodesic equation to show that, for an object in radial free-fall towards a black hole, $\frac{dt}{d\tau} = \frac{K}{1-\frac{R_S}{r}}$ and $\left(\frac{1}{c}\frac{dr}{d\tau}\right)^2 = K^2 - \left(1 - \frac{R_S}{r}\right)$, where $K = \text{constant}$

- If the object starts from rest at $r = \infty$, then $K = 1$

- We deduce $\frac{1}{c}\frac{dr}{d\tau} = -\sqrt{\frac{R_S}{r}}$, or **plunge time** $\boldsymbol{\Delta\tau = \frac{2r^{3/2}}{3c{R_s}^{1/2}}}$

Radial plunge into a black hole

- The co-ordinate time along the path is $\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = \frac{\sqrt{r/R_S}}{c(1-r/R_S)}$

- Can solve to obtain $t = \frac{R_S}{c}\left\{2\left(\frac{r}{R_S}\right)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{r}{R_S}\right)^{\frac{3}{2}} + \ln\left|\frac{\sqrt{r/R_S}+1}{\sqrt{r/R_S}-1}\right|\right\}$

- $t = \infty$ at $r = R_s$??

- **The proper time interval for the object to reach $r = 0$ (or $r = R_s$) is finite, but the co-ordinate time interval is infinite!!**

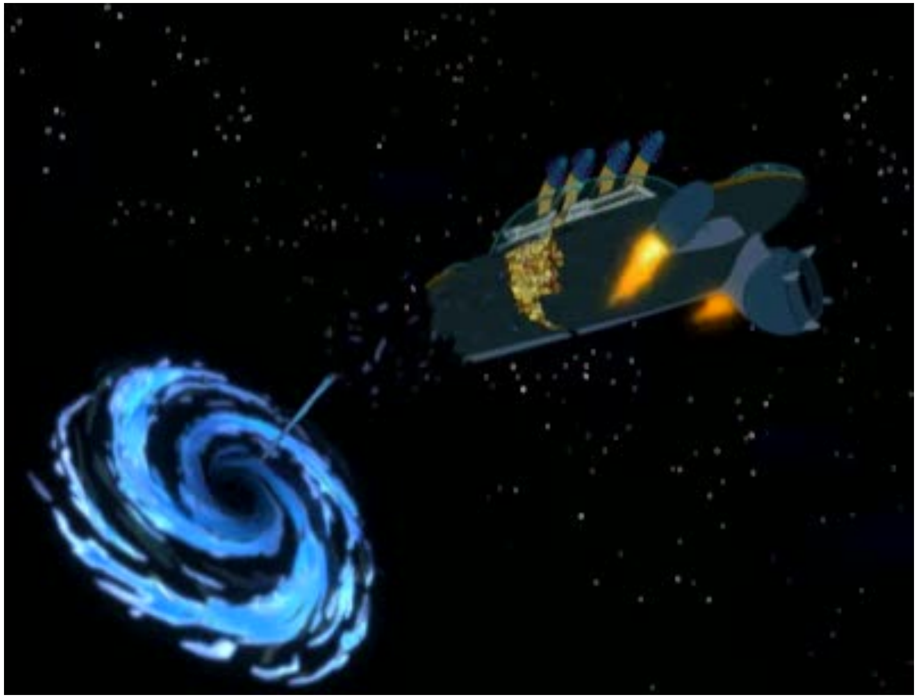
- The definition of t in the Schwarzschild metric has a problem!

Falling into a black hole

Let's compare the different perspectives of a spacecraft plunging into a black hole of …

- *spacecraft travellers* (e.g. students!)
- *a distant observer* (e.g. professor sipping cocktails!)

The spacecraft travellers are sending back regular light signals in their frame



http://scienceblogs.com/startswithabang/2009/11/20/believe-it-or-not-a-black-hole/

Falling into a black hole

- **From the perspective of the spacecraft travellers …**
- The clocks on the spacecraft are using proper time τ (i.e. are present at all events)
- The travellers notice no singularity at $r = R_s$, and complete their plunge from $r = R_0$ to $r = 0$ in time $\tau = \frac{2{R_0}^{3/2}}{3c{R_s}^{1/2}}$
- As r decreases, the *tidal gravitational force (stretching force) on the spacecraft increases*
- At $r = R_s$, *the proper acceleration that would be required to hold a static position becomes infinite*

Falling into a black hole

- **From the perspective of the distant observer …**
- The co-ordinate time interval between the light signals increases, since $dt = d\tau/\sqrt{1 - R_s/r}$ and $d\tau =$ constant
- The spacecraft will seem to slow, and stop as $r \rightarrow R_s$
- The light signals from the spacecraft are being gravitationally redshifted as $f_\infty/f_r = \sqrt{1 - R_s/r}$
- The image of the spacecraft redshifts and fades from view
- The distant observer is effectively aging more quickly relative to the spacecraft travellers

The photon sphere

- Consider a light ray in a circular orbit around the black hole, such that $dr = 0$. We choose the orbit such that $\theta = 90°$
- $ds = 0$ implies that $\frac{1}{c}\frac{d\phi}{dt} = \frac{\sqrt{1-R_S/r}}{r}$
- The $\mu = r$ geodesic equation implies that $\frac{1}{c}\frac{d\phi}{dt} = \sqrt{\frac{R_S}{2r^3}}$
- We hence find that **light has a circular orbit around a black hole at the photon sphere $\boldsymbol{r = \frac{3}{2}R_s}$**
- *Any circular orbit closer than this is space-like (impossible)*

Nature of the singularity

- A co-ordinate singularity is a place where *the chosen set of co-ordinates does not describe the geometry properly*

- An example is at the North Pole of a spherical co-ordinate system, where all values $0 < \phi < 2\pi$ correspond to a single point in space!

- Similarly, **the point $\boldsymbol{r = R_s}$ of the Schwarzschild geometry is a co-ordinate singularity. $\boldsymbol{r = 0}$ is a true singularity**

- We can transform to another radial co-ordinate system in which this singularity is removed