Class 7: Geodesics

In this class we will discuss the equation of a geodesic in a curved space, how particles and light rays move in a curved space-time, and how this motion connects to Newton's Laws

Class 7: Geodesics

At the end of this session you should be able to …

- … recognize the **geodesic equation**, which connects the motion of particles to the space-time metric, using Christoffel symbols
- … calculate the **Christoffel symbols** in some simple cases, such as in 2D spaces or the weak-field limit
- … recognize that different parameters can be used to describe the world line of **particles and light rays**
- … understand **how GR connects to Newton's Laws** for weak fields, and how the time co-ordinate behaves in this limit

Geodesics

- A fundamental question for General Relativity to answer is, ***how does an object move in a gravitational field?***

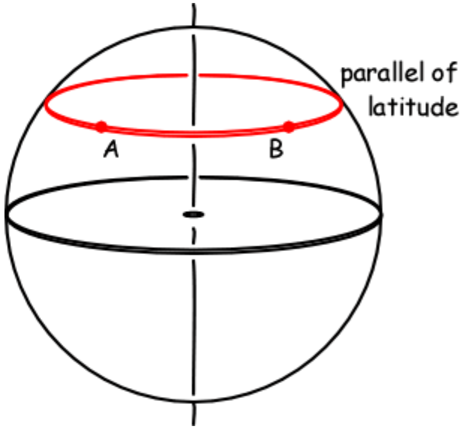
https://mathspig.wordpress.com/2014/01/23/2-one-rule-aerial-skiers-cannot-break/

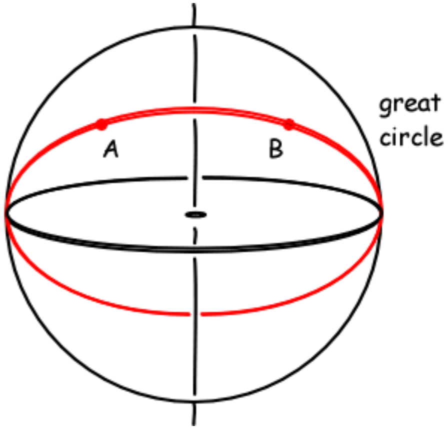
- How can we find an object's **world line** $x^{\mu}(\tau)$ in terms of its proper time τ, when freely falling in the Earth's frame x^{μ}?

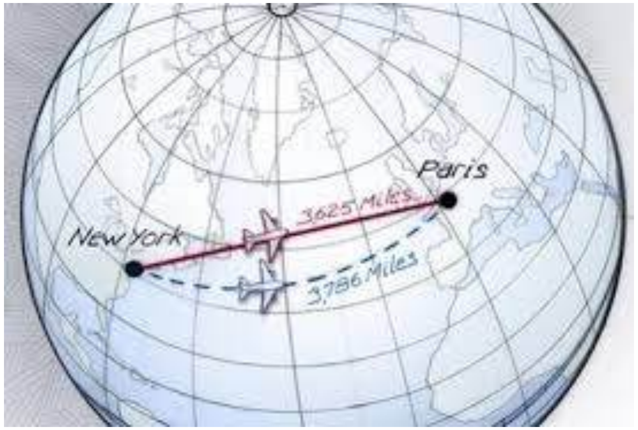
Geodesics

- A "straight line on a curved surface" is called a **geodesic**, which *minimizes the distance* between 2 points

- e.g., geodesics on a spherical surface are "great circles"





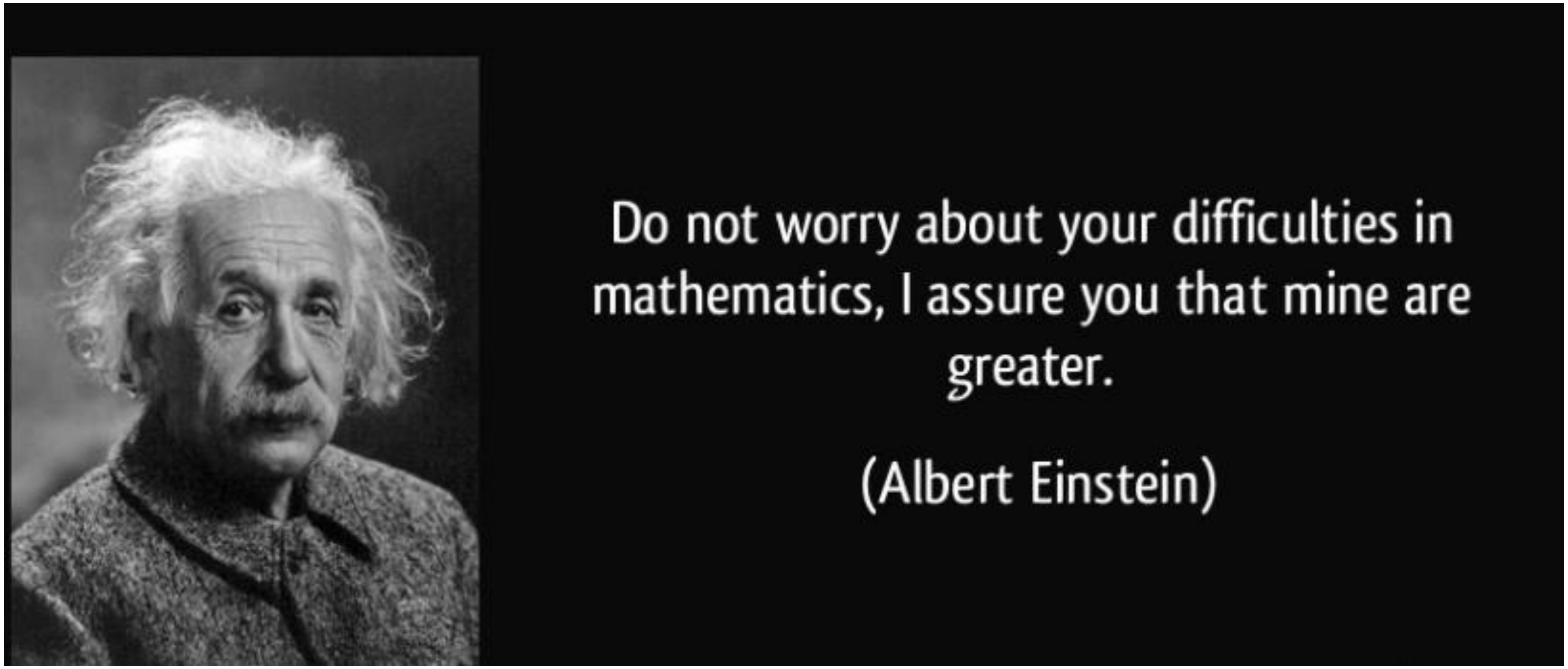


- In GR, **objects travel on a geodesic in curved space-time**, which *extremizes the proper time* between 2 points

Geodesics

- The same mathematics hence describes **both** the geometry of curved spaces **and** the geometry of space-time

- This maths was Einstein's biggest challenge in developing GR!



https://www.pinterest.com.au/pin/407083253789581007/

Geodesic equation

- Objects move on a path which extremizes the proper time
- Since $ds^2 = -c^2 d\tau^2$, the proper time along a path is given in terms of the metric by $\frac{1}{c}\int\sqrt{-ds^2} = \frac{1}{c}\int\sqrt{-g_{\mu\nu}\,dx^\mu\,dx^\nu}$
- We can mathematically find the path $x^\mu(\tau)$ that *extremizes the value of this integral.* The result is the **geodesic equation**:

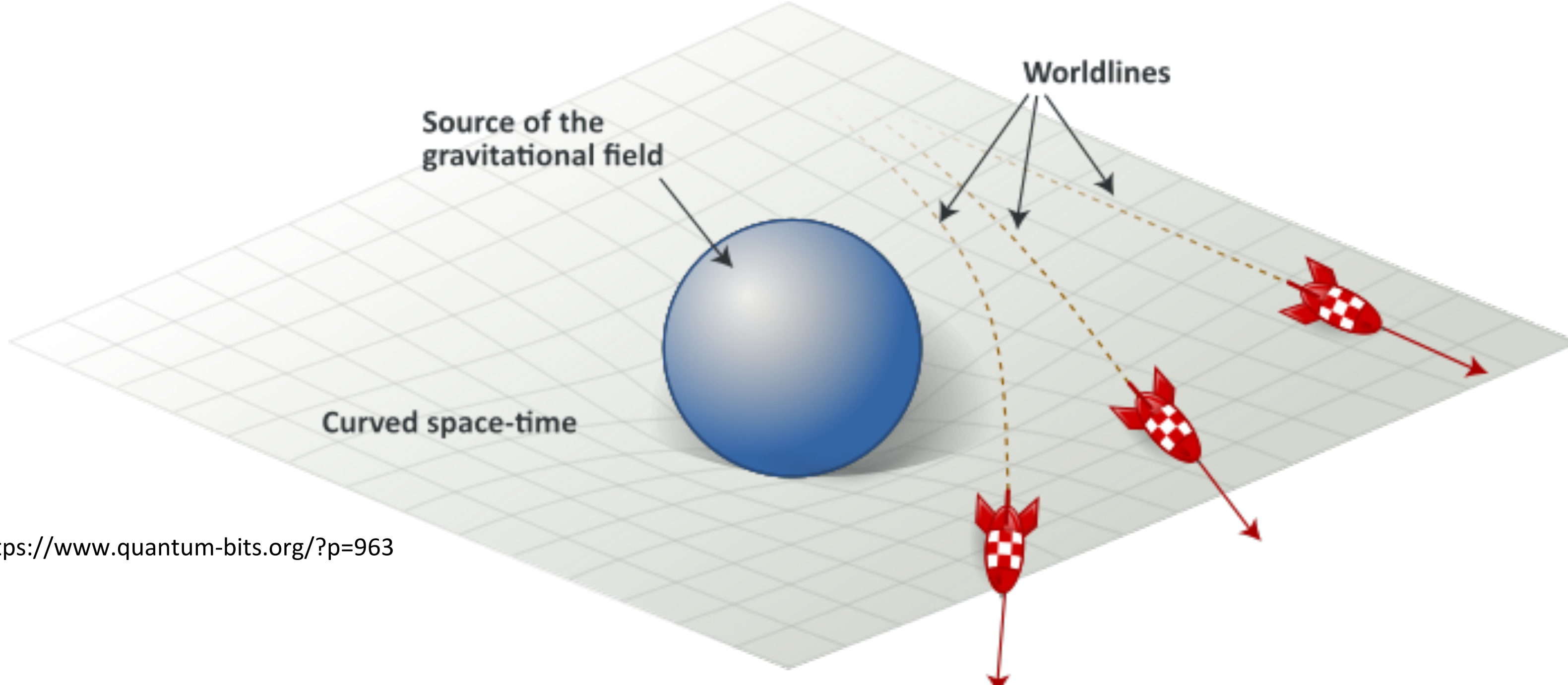
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$$

Geodesic equation

- What's the physical meaning of $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$?

- It's the **equation of motion of freely-falling particles in a curved space-time**. Particles travelling on a geodesic feel no forces



https://www.quantum-bits.org/?p=963

Geodesic equation

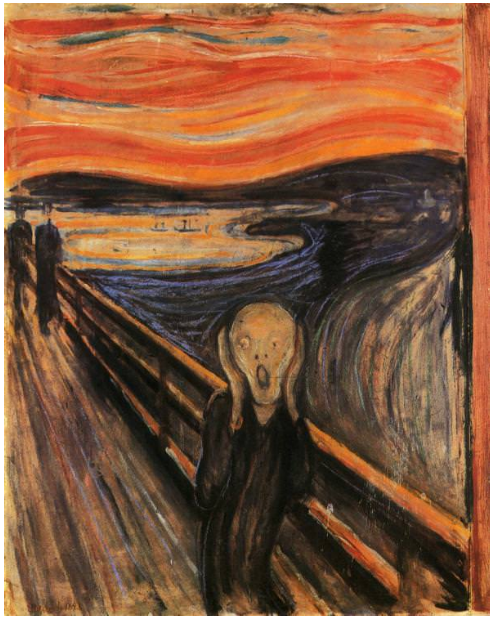
- What's the physical meaning of $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$?
- The indices κ and λ are summed over, hence this relation represents 4 differential equations, which can be solved to determine the path of a particle through space-time, $x^\mu(\tau)$
- If $\Gamma^\mu_{\kappa\lambda} = 0$, then the particle has zero acceleration $\frac{d^2x^\mu}{d\tau^2} = 0$ and is moving uniformly in an inertial frame – so $\boldsymbol{\Gamma = 0}$ **in the absence of gravity**
- Hence $\boldsymbol{\Gamma^\mu_{\kappa\lambda}}$ **represents a "force" due to gravity**, which is curving the path of the particle through space-time

Christoffel symbols

- The values of $\Gamma^{\mu}_{\kappa\lambda}$ are determined by the space-time metric $g_{\mu\nu}$, as $\Gamma^{\mu}_{\kappa\lambda} = \frac{1}{2} g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$
- This object $\Gamma^{\mu}_{\kappa\lambda}$ is so important that it has a name – the **Christoffel symbols**
- **What is $g^{\mu\nu}$?** If $g_{\mu\nu}$ is written as a matrix, then $g^{\mu\nu}$ is the *inverse of the matrix*
- $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{dy}, \frac{\partial}{dz}\right)$ is differentiating the metric elements
- The index ν is summed over

Christoffel symbols

- The problem is that, since each index can take on 4 values, $\Gamma^{\mu}_{\kappa\lambda}$ consists of $4{\times}4{\times}4 = 64$ different functions in general!

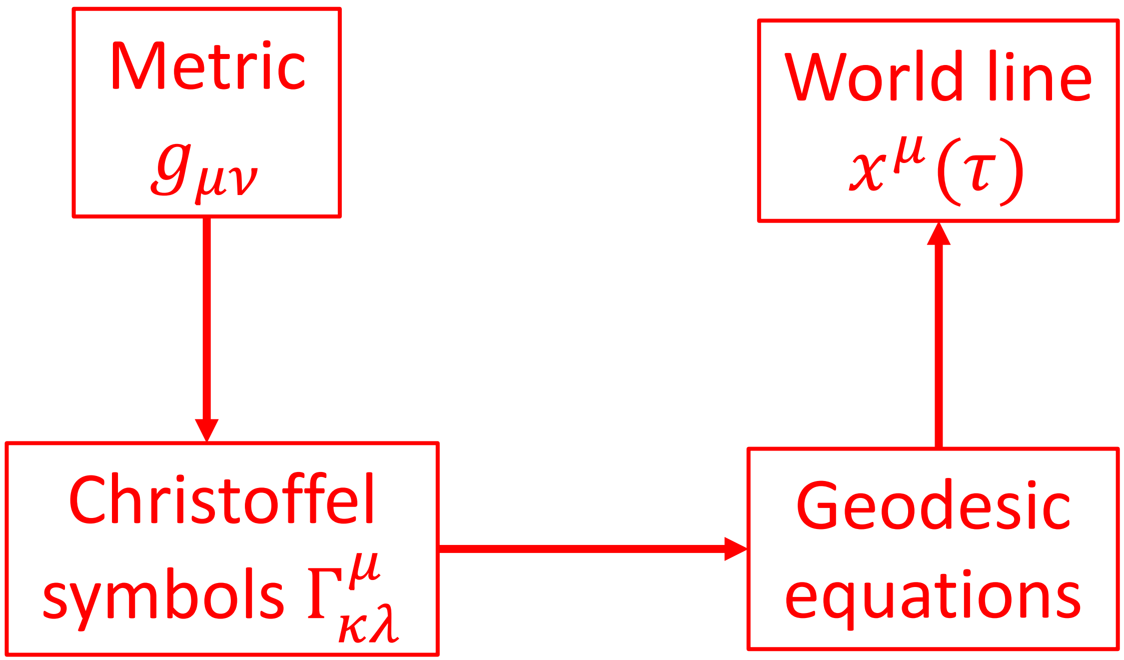


https://www.edvardmunch.org/the-scream.jsp

- However, it is easier for the special cases we'll consider

Christoffel symbols

- **Space-time curvature tells matter how to move**



Christoffel symbols

- There is a calculational trick which can make it easier to determine the Christoffel symbols in some cases …
- The geodesic equation is: $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$
- A mathematically equivalent version of the geodesic equation is: $g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \left(\partial_\lambda g_{\mu\kappa} - \frac{1}{2}\partial_\mu g_{\kappa\lambda}\right)\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0$
- Sometimes it's easier to evaluate the second equation and determine $\Gamma^\mu_{\kappa\lambda}$ by comparison with the first equation, rather than to evaluate $\Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\nu}(\partial_\lambda g_{\nu\kappa} + \partial_\kappa g_{\lambda\nu} - \partial_\nu g_{\kappa\lambda})$

Motion in a weak field

- An important special case is a particle **moving slowly in a weak, static gravitational field**

- In this case, the space-time metric can be written in the form $g_{\mu\nu}(x^i) = \eta_{\mu\nu} + h_{\mu\nu}(x^i)$, where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the Minkowski metric, and $|h_{\mu\nu}| \ll 1$ is a small perturbation which depends only on spatial co-ordinates x^i, not time

- For a slow-moving particle, $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}$ where $i = \{1,2,3\}$

Motion in a weak field

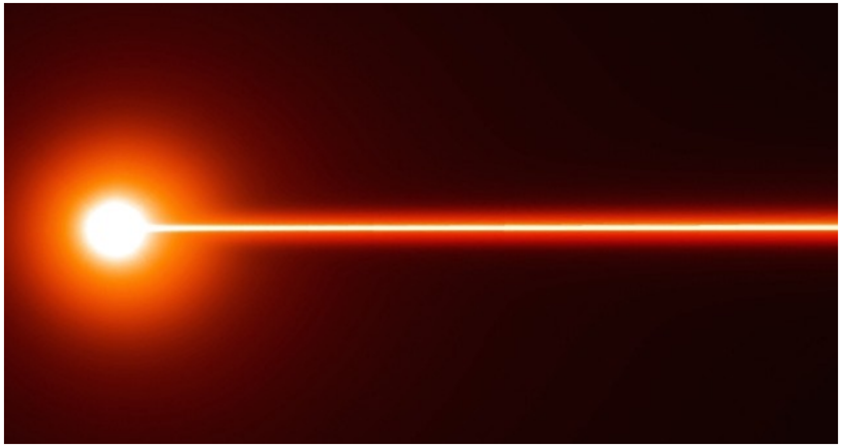
- Using these approximations, the geodesic equations imply that $\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\frac{\partial h_{tt}}{\partial x_i}$

- We can compare this relation to Newton's laws in a gravitational potential ϕ, $\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi(\vec{x})$

- We hence deduce that $g_{tt} = -1 - 2\phi/c^2$

- For a clock at rest in a weak field, co-ordinate and proper time are related by $d\tau = \sqrt{-g_{tt}}\,dt = \sqrt{1 + 2\phi/c^2}\,dt$

Geodesics for light rays



- For a light ray, there is a subtlety which wrecks our previous derivation – $ds = d\tau = 0$. We cannot describe the path as a function of τ, since $\tau = 0$ always! (it's a "null geodesic")
- We need to parameterize the world line by a different co-ordinate called the "affine parameter", $x^{\mu}(p)$
- We end up with the same equation, $\frac{d^2 x^{\mu}}{dp^2} + \Gamma^{\mu}_{\kappa\lambda} \frac{dx^{\kappa}}{dp} \frac{dx^{\lambda}}{dp} = 0$

Geodesics for light rays

- It's useful to consider the **wavevector** $k^\mu = \frac{dx^\mu}{dp}$, in terms of which $\frac{dk^\mu}{dp} + \Gamma^\mu_{\kappa\lambda}\, k^\kappa\, k^\lambda = 0$

- In physical terms, the wavevector k^μ gives the frequency (k^0) and direction of motion (k^i) of the light ray

- In the absence of a gravitational field, $\Gamma = 0$, hence $\frac{dk^\mu}{dp} = 0$, hence $k^\mu =$ constant

- The gravitational field "bends" the light ray according to Γ, changing its direction of travel