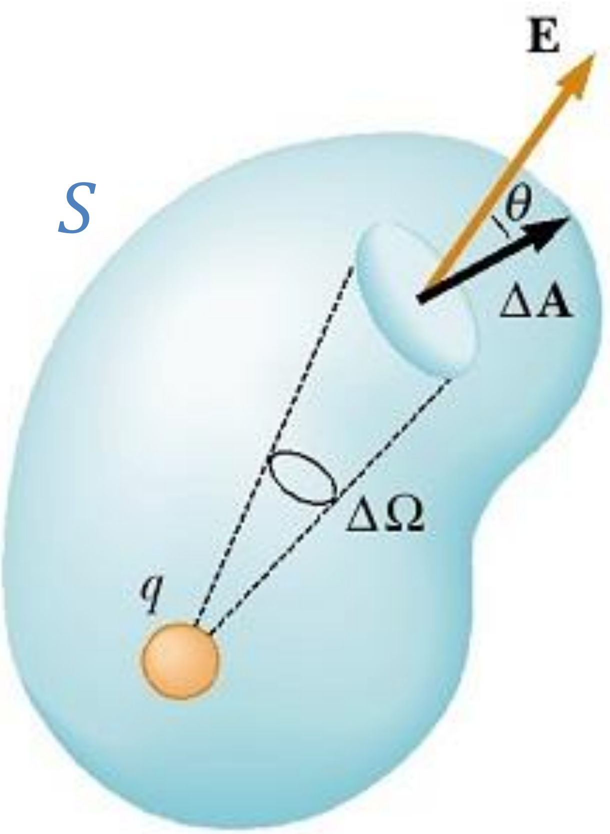
Class 4 : Maxwell's Equations for Electrostatics

- Concept of charge density
- Maxwell's 1st and 2nd Equations
- Physical interpretation of divergence and curl
- How do we check whether a given vector field could be an electric field?

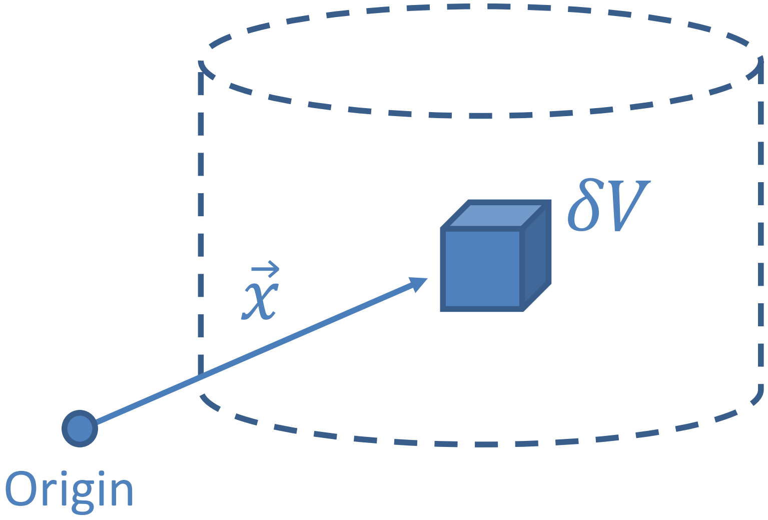
Recap



- The electric field $\vec{E}$ around a charge distribution can be determined by **Gauss's Law** $\int \vec{E}.d\vec{A} = Q_{enclosed}/\varepsilon_0$
- The $\vec{E}$-field may be described as the gradient of an **electrostatic potential** V, where $\vec{E} = -\vec{\nabla}V$
- The **potential difference** is the work done in moving a unit charge between 2 points, $\Delta V = -\int \vec{E}.d\vec{l}$

Charge density

- We often want to merge point charges together, and think of a **continuous distribution** of charge
- We define the **charge density** at position $\vec{x}$, or charge per unit volume, as $\rho(\vec{x})$



- Consider a volume element δV
- Charge in this volume element $\delta Q = \rho\, \delta V$
- Total charge $Q = \int \rho\, \delta V$

Vector divergence

- We can apply $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ to a vector function $\vec{E} = (E_x, E_y, E_z)$ using either a **dot product** or a **cross product**

- We will first consider the dot product. Using the usual dot product rule, we have $\vec{\nabla}.\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

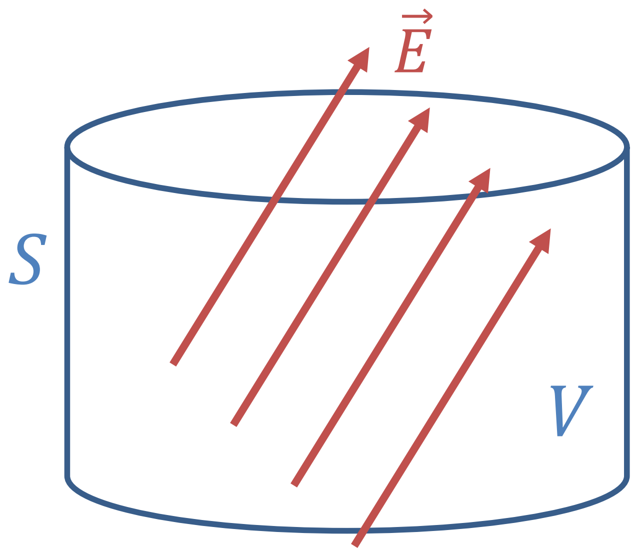
- This is known as the **divergence of a vector field** (we will explain why shortly!)

- We will meet the cross product version soon!

Divergence theorem

- A powerful result of vector calculus, which links a surface integral with the divergence, is the **vector divergence theorem**

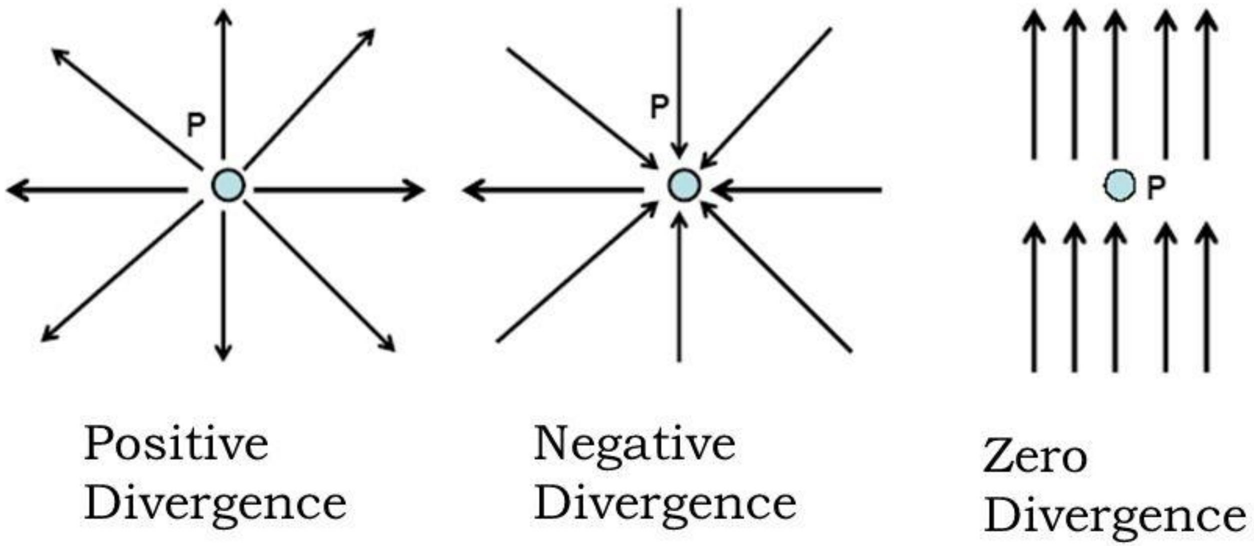
- If S is a closed surface around a volume V, then for any vector field $\vec{E}$ we can say:



$$\int \vec{E}.d\vec{A} = \int \left(\vec{\nabla}.\vec{E}\right) dV$$

Physical meaning of divergence

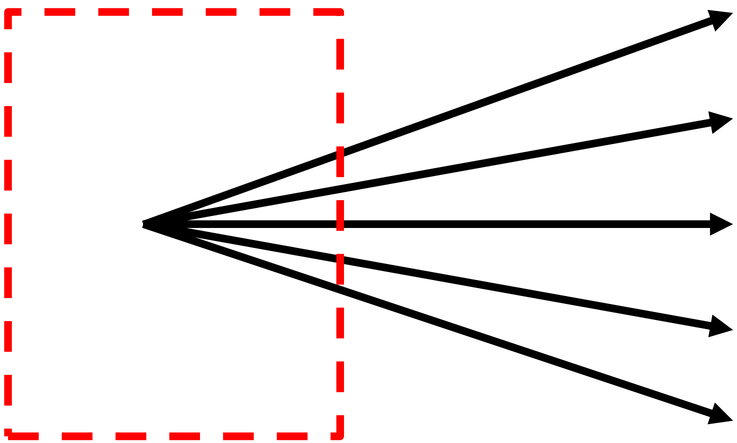
- The divergence of a vector field at a point P measures **how much it is flowing out of, or into, that point**



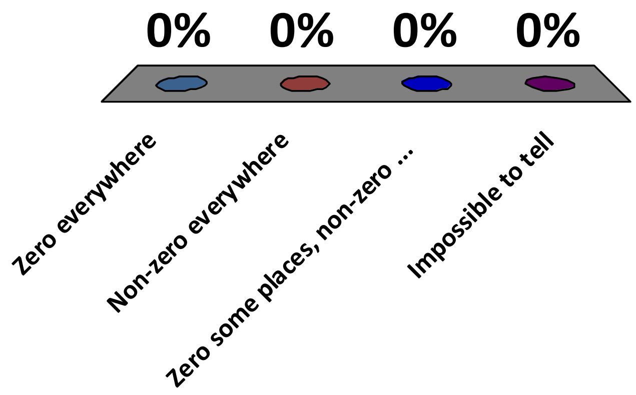
- We can see this from $\int \vec{E}.d\vec{A} = \int\left(\vec{\nabla}.\vec{E}\right) dV$ - the divergence $\vec{\nabla}.\vec{E}$ creates an outward flux $\int \vec{E}.d\vec{A}$

Clicker question

Consider the following vector field.
What is the divergence in the box?



A. Zero everywhere
B. Non-zero everywhere
C. Zero some places, non-zero other places
D. Impossible to tell



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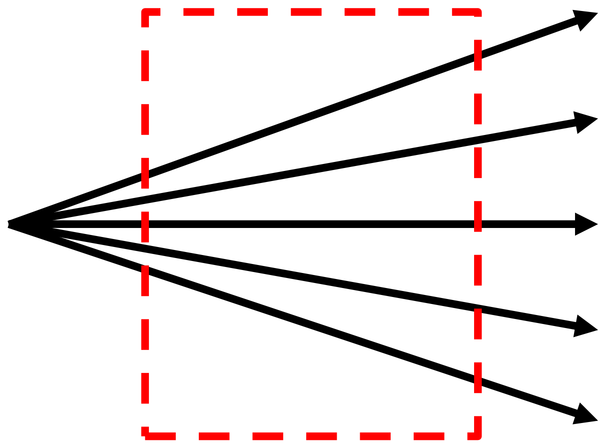

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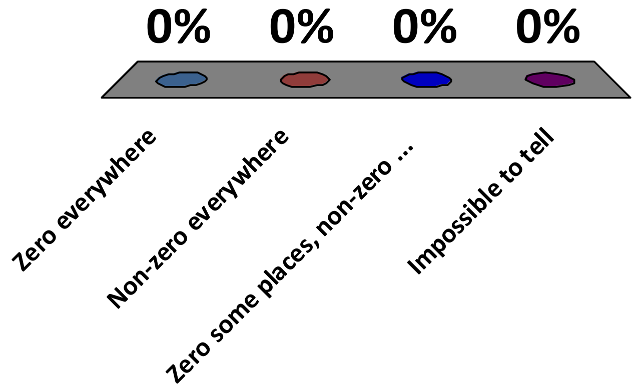


Maxwell's 1st Equation

- Apply the vector divergence theorem to Gauss's Law:

$$\int \vec{E} \,.\, d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

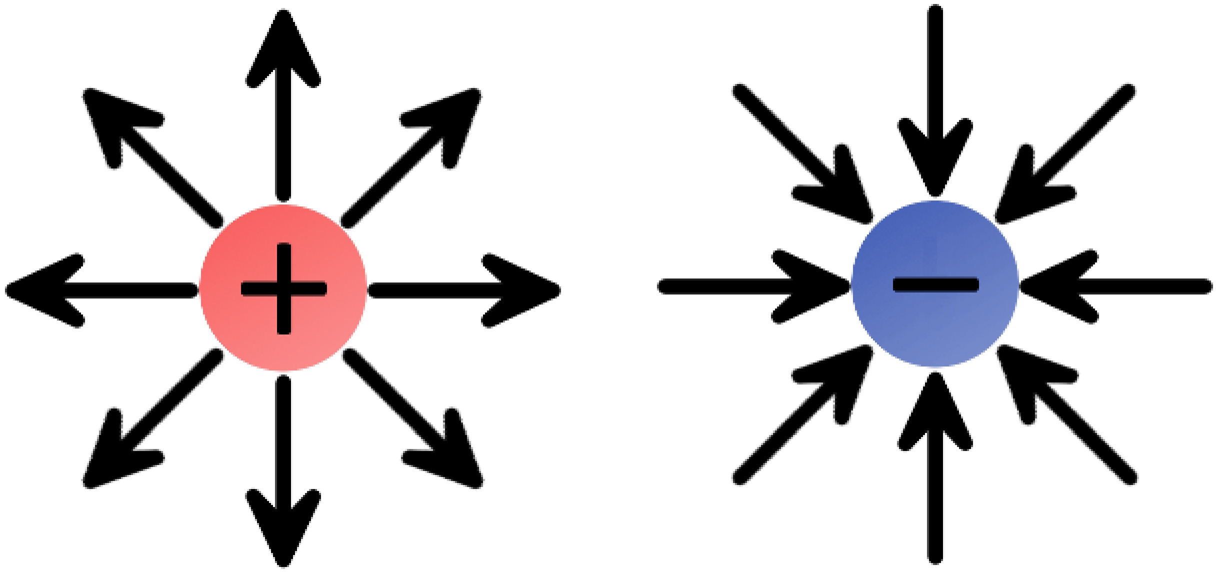
$$\int (\vec{\nabla}.\vec{E})\, dV \qquad\qquad \int \rho \, dV$$

- Re-arranging: $\int \left(\vec{\nabla}.\vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$ (for any volume)

- This implies: $\boldsymbol{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}}$

Maxwell's 1st Equation

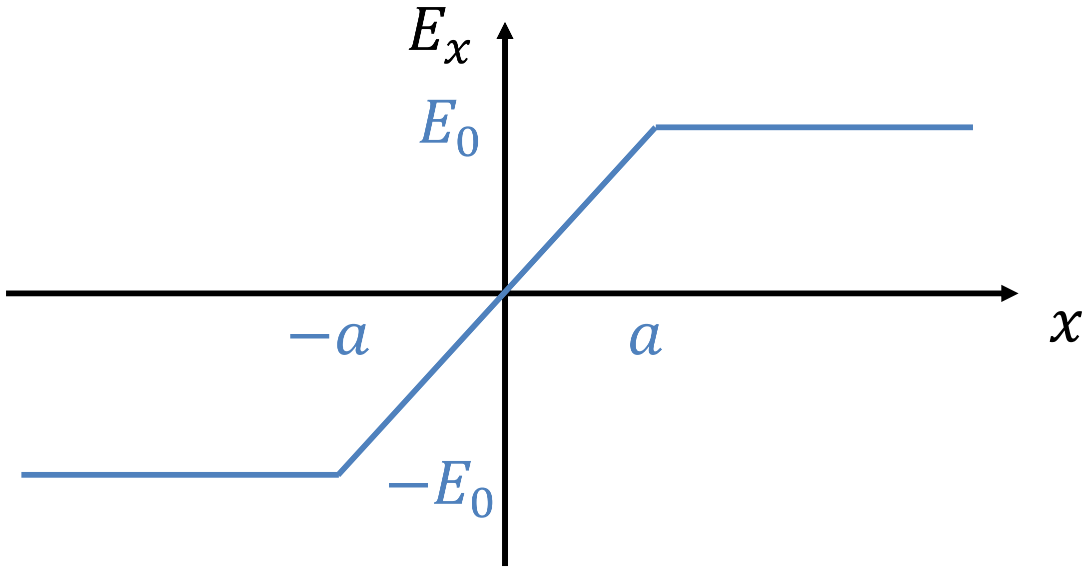
- Maxwell's beautiful equation $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ is describing mathematically how **electric field lines flow away from positive charges and toward negative charges**



Please note in workbook

Maxwell's 1st Equation

- Consider the following electric field:



- What is the corresponding charge density and electrostatic potential?

Vector curl

- We have already met the **divergence** of a vector field, $\vec{\nabla}.\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ where $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

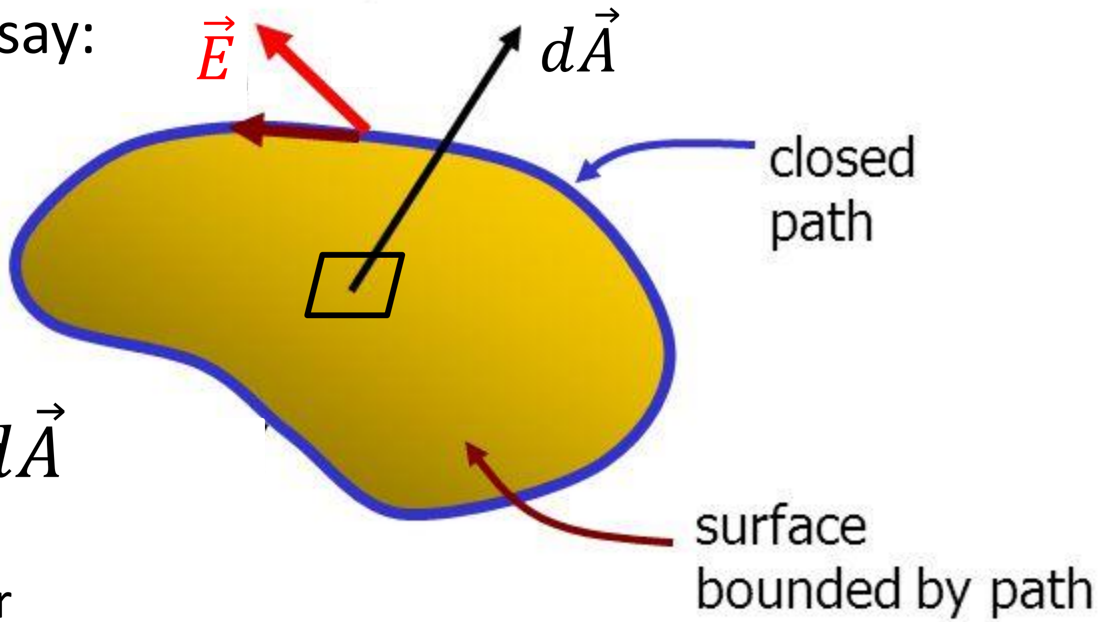
- We now consider applying the $\vec{\nabla}$ operator to a vector using the cross product $\times$, which is known as a **vector curl** $\vec{\nabla} \times \vec{E}$

- The curl is evaluated as a **cross product** and is a vector, e.g.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Stokes' Theorem

- A fundamental result in vector calculus called **Stokes' Theorem** relates the vector curl to a line integral
- If S is any surface bounded by a closed loop L, then for any vector field $\vec{E}$ we can say:



$$\oint \vec{E}.d\vec{l} = \int (\vec{\nabla} \times \vec{E}).d\vec{A}$$

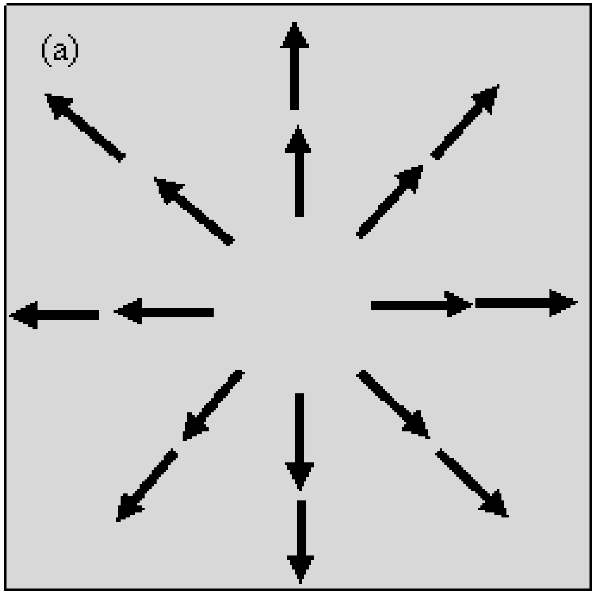
Integral around closed loop

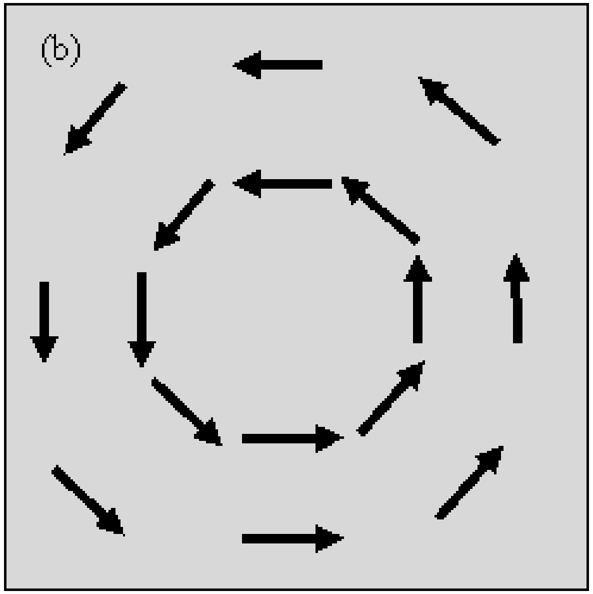
Integral over surface

Physical meaning of vector curl

- The **divergence** ($\vec{\nabla}.$) describes the amount of *outflow or inflow* of a vector field at a point

- The **curl** ($\vec{\nabla}\times$) describes the *circulation* of a vector field about a point (and the axis of the circulation)



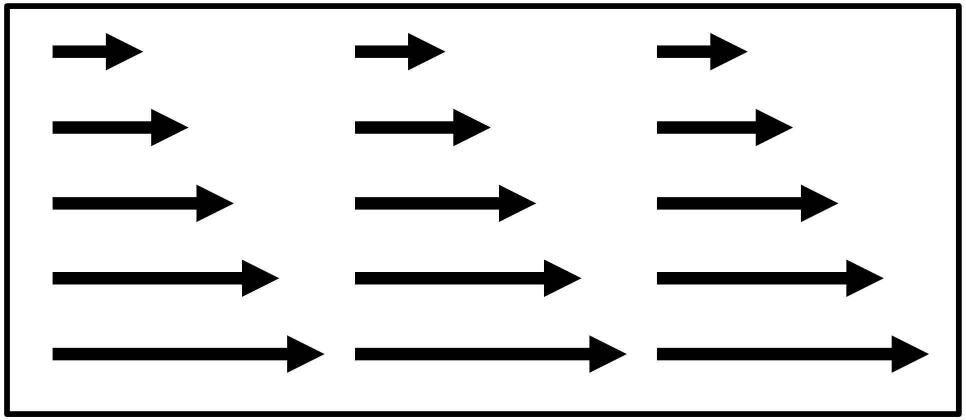


Physical meaning of vector curl

- The **divergence** $(\vec{\nabla}.)$ describes the amount of *outflow or inflow* of a vector field at a point
- The **curl** $(\vec{\nabla}\times)$ describes the *circulation* of a vector field about a point (and the axis of the circulation)
- We can see this is the case using Stokes' Theorem : if $\vec{\nabla}\times\vec{E} > 0$, then there will be a non-zero integral around a closed loop, $\oint \vec{E}.d\vec{l} > 0$, corresponding to a circulation

Clicker question

Consider the vector field shown.
What can we say about its curl?



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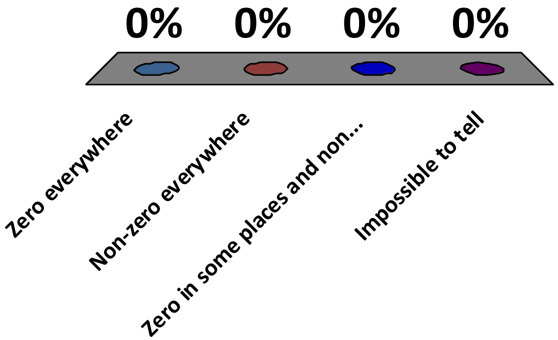


Maxwell's 2nd equation

- We can use the above results to deduce **Maxwell's 2nd equation** (in electrostatics)

- If we move an electric charge in a closed loop we will do zero work : $\oint \vec{E}.d\vec{l} = 0$

- Using Stokes' Theorem, this implies that for any surface in an electrostatic field, $\int \left(\vec{\nabla} \times \vec{E}\right).d\vec{A} = 0$

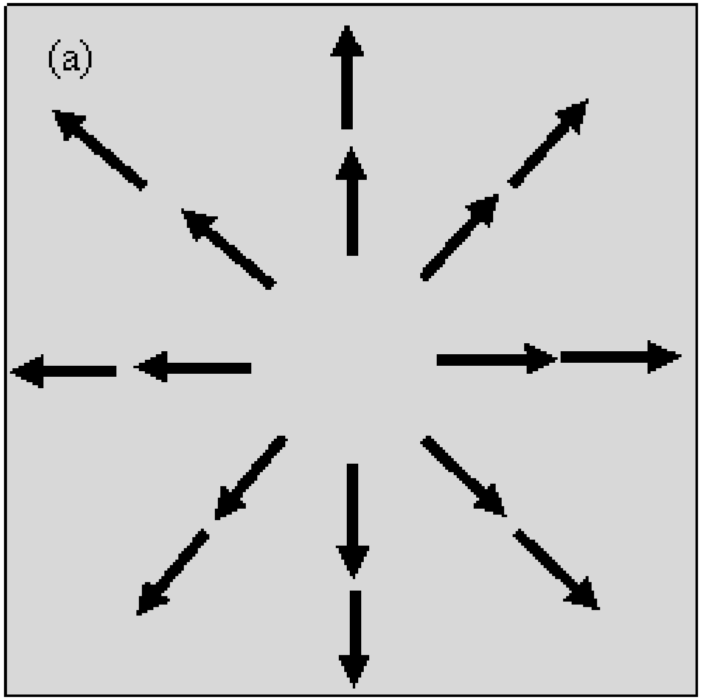
- Since this is true for any surface, then $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = \vec{\mathbf{0}}$

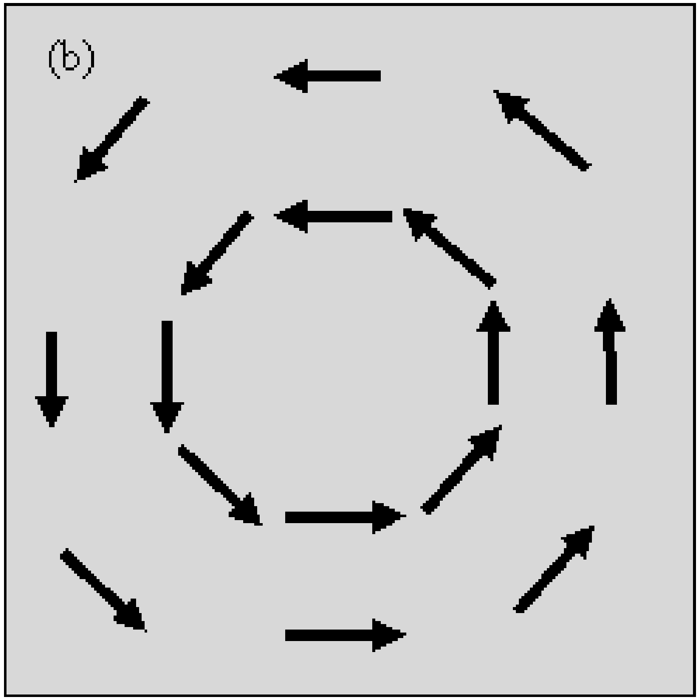
Maxwell's 2nd equation

- We could have deduced the same result using the fact that the electric field can be expressed as a potential, $\vec{E} = -\vec{\nabla}V$
- Applying the curl operator to both sides, we find $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla}V = \vec{0}$ (which can be shown by multiplying out the components of $\vec{\nabla} \times \vec{\nabla}V$)
- $\boldsymbol{\vec{\nabla} \times \vec{E} = \vec{0}}$ **is Maxwell's 2nd equation as applied to electrostatics**
- (It must be modified for time-varying fields as we will describe in later lectures)

Maxwell’s 2nd equation

- In physical terms, the fact that $\vec{\nabla} \times \vec{E} = \vec{0}$ is describing that electric field lines **do not circulate into loops** – they always look like diagram (a) below, not diagram (b)

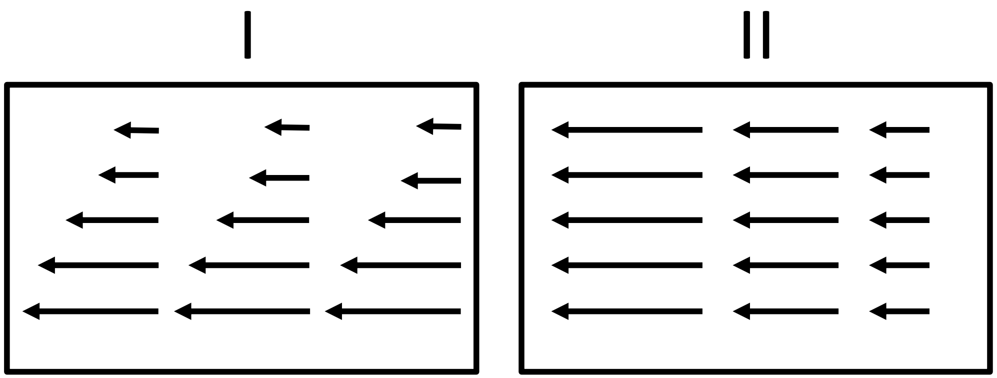




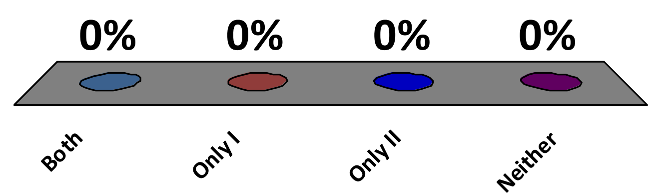
Please note in workbook

Clicker question

Which of the following could be an electrostatic field in the region shown?



A. Both
B. Only I
C. Only II
D. Neither



Summary

- Vector calculus is a powerful mathematical tool based on the vector operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
- **Maxwell's 1st Equation** expresses Gauss's Law in the differential form $\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{E}} = \frac{\boldsymbol{\rho}}{\boldsymbol{\varepsilon_0}}$ *(field lines end on charges)*
- **Maxwell's 2nd Equation** for electrostatics is $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = \vec{\boldsymbol{0}}$ *(field lines do not circulate)*
- This is equivalent to defining an electrostatic potential V as $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\boldsymbol{V}$