# Class 3 : Electrostatic Potential

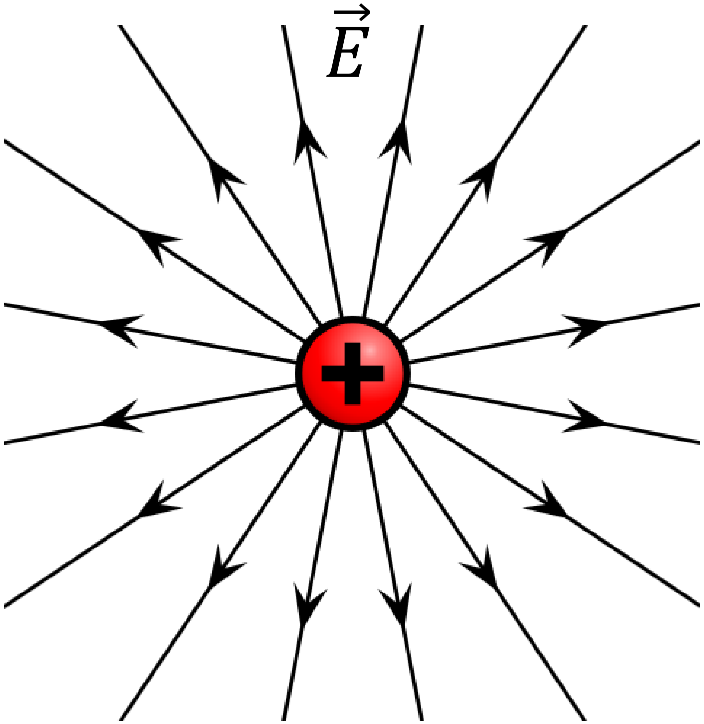
- What is electrostatic potential?
- How do we determine it from charges?
- What is its relation to electric field?
- The importance of path-independence
- The electrostatic potential energy of a system of charges

# Recap

- An **electric field** $\vec{E}$ is set up in the region of space around charges

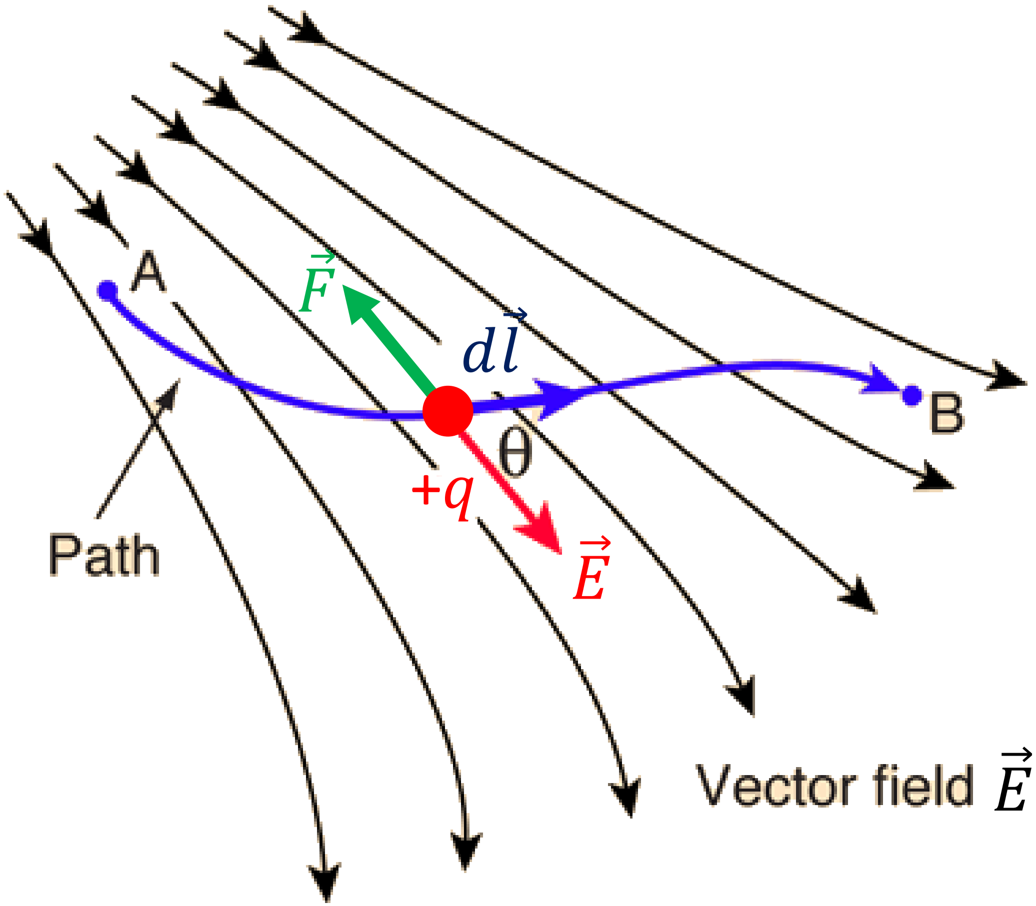
- The electric field can be computed using **Gauss's Law**

- A charge $q$ placed in the electric field will feel a force $\vec{F} = q\vec{E}$



# Electrostatic potential

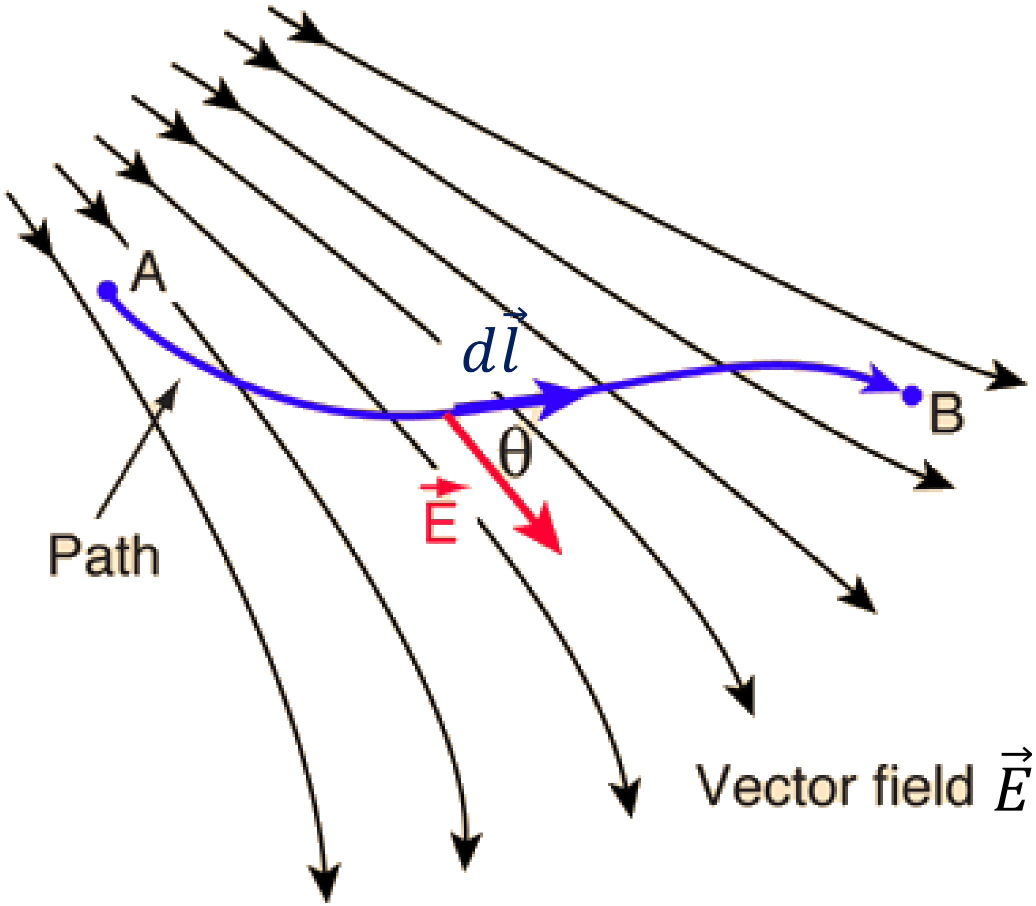
Consider the work done in moving a charge $+q$ between two points $A$ and $B$ of an electric field $\vec{E}$




- The force from the field is $q\vec{E}$
- The force doing the work is $\vec{F} = -q\vec{E}$
- Along a curve joining $A$ and $B$, sum up the elements $\vec{F}.d\vec{l} = F\ dl\ \cos\theta$
- **Work** $= -q\int_A^B \vec{E}.d\vec{l}$

# Line integral

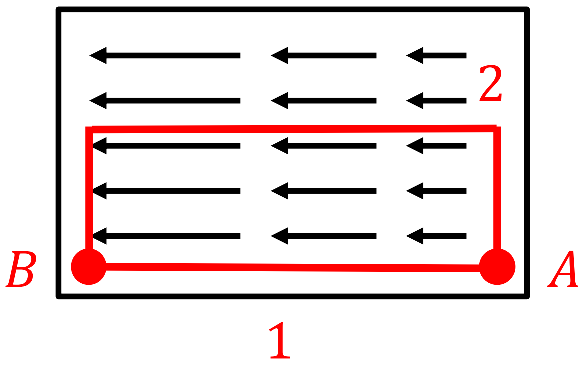
- This uses the concept of a **line integral of a vector field**




- We split the line joining points $A$ and $B$ into line elements $d\vec{l}$
- A line integral sums up the dot products $\vec{E}.d\vec{l}$ along the line
- It is written $\int_A^B \vec{E}.d\vec{l}$

# Clicker question

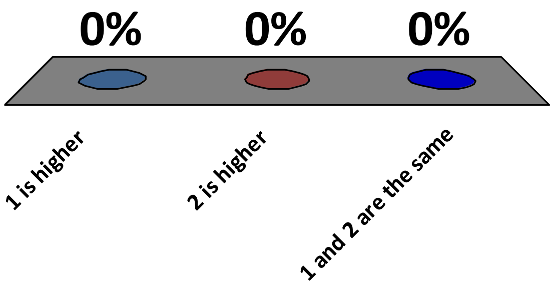
Here is a vector field $\vec{E}$. Which path between $A$ and $B$ has the higher value of $\int \vec{E}.d\vec{l}$?




A. 1 is higher

B. 2 is higher

C. 1 and 2 are the same



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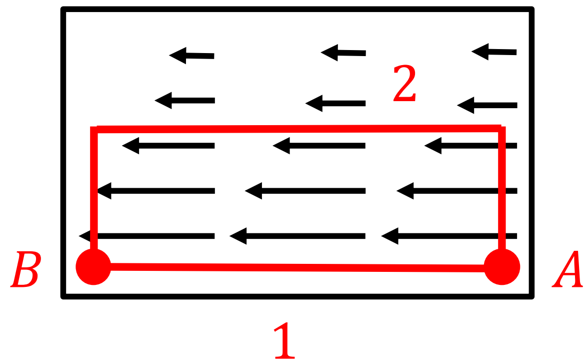


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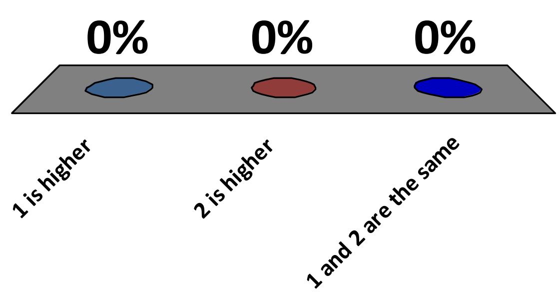
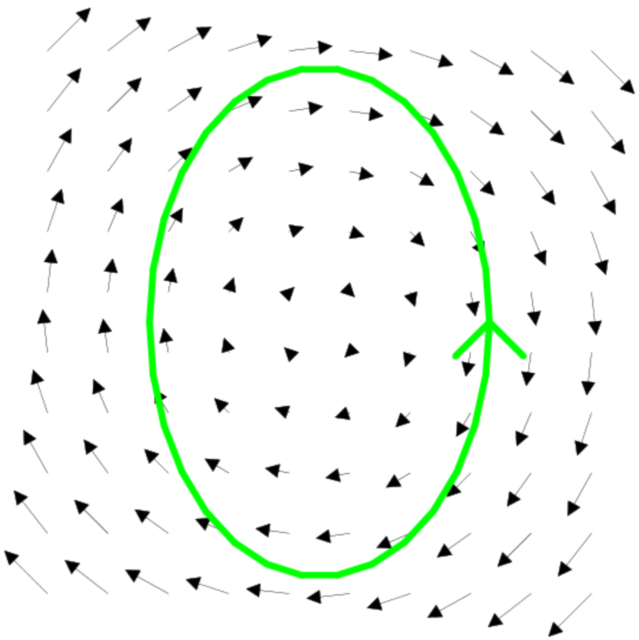


# Line integral

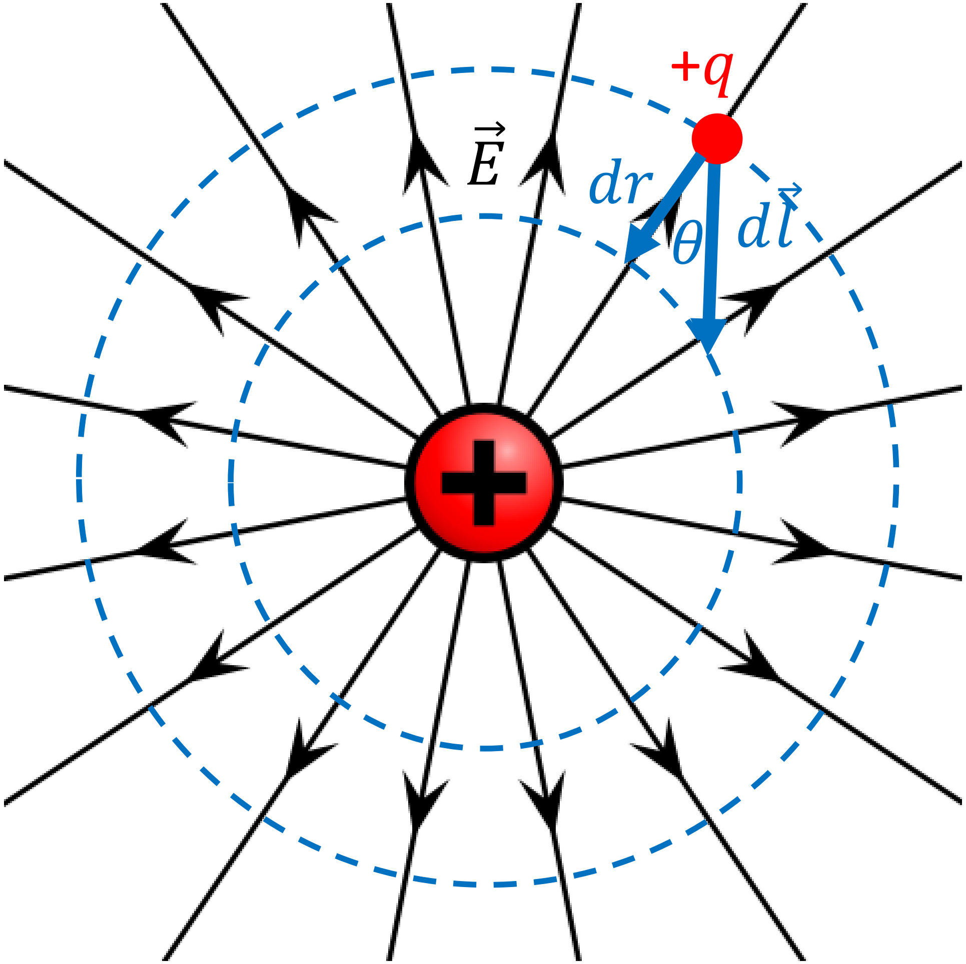
- Sometimes our line integral of a vector field $\vec{E}$ may be around a **closed loop**

- In this case, we write the integral as $\oint \vec{E}.d\vec{l}$



# Electrostatic potential

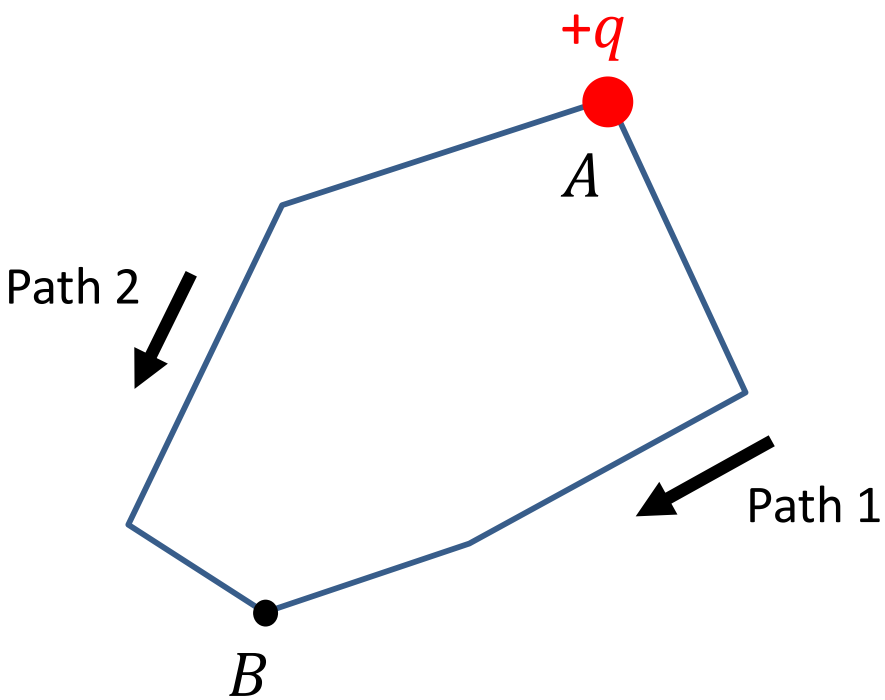
The work done has the special property that it is **independent of the path taken** between $A$ and $B$




- Consider the work done in moving a test charge $+q$ a displacement $d\vec{l}$ in the vicinity of another charge
- Work done $= \vec{F}.d\vec{l} = -q\vec{E}.d\vec{l} = -q\,E\,dl\,\cos\theta = -q\,E\,dr$
- This is independent of $\theta$! So *the total work done just depends on the radii, not the path taken*

# Electrostatic potential

The work done has the special property that it is **independent of the path taken** between $A$ and $B$




- This has an important implication!
- If $-q\int_{Path\ 1}\vec{E}.d\vec{l} = -q\int_{Path\ 2}\vec{E}.d\vec{l}$ it follows that …
- $\oint \vec{\boldsymbol{E}}.\boldsymbol{d}\vec{\boldsymbol{l}} = \boldsymbol{0}$
- Moving a particle around a closed path requires no work

# Vector gradient operator

- In this course we will occasionally need to use the **beautiful language of vector calculus**

- Vector calculus is expressed using the special symbol $\vec{\nabla}$

- $\vec{\nabla}$ is a vector operator with components $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where these are derivatives with respect to each co-ordinate

- If we apply this operator to a scalar function $f(x, y, z)$, we will obtain a vector $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ which gives the **vector gradient of the function** in each direction

# Electrostatic potential

- We define the **electrostatic potential difference** between two points $A$ and $B$ as *the work done in moving a unit charge* $(q = 1)$ *between those points*

Please note in workbook

- Hence $\Delta \boldsymbol{V} = -\int_{A}^{B} \vec{\boldsymbol{E}}.\boldsymbol{d}\vec{\boldsymbol{l}}$

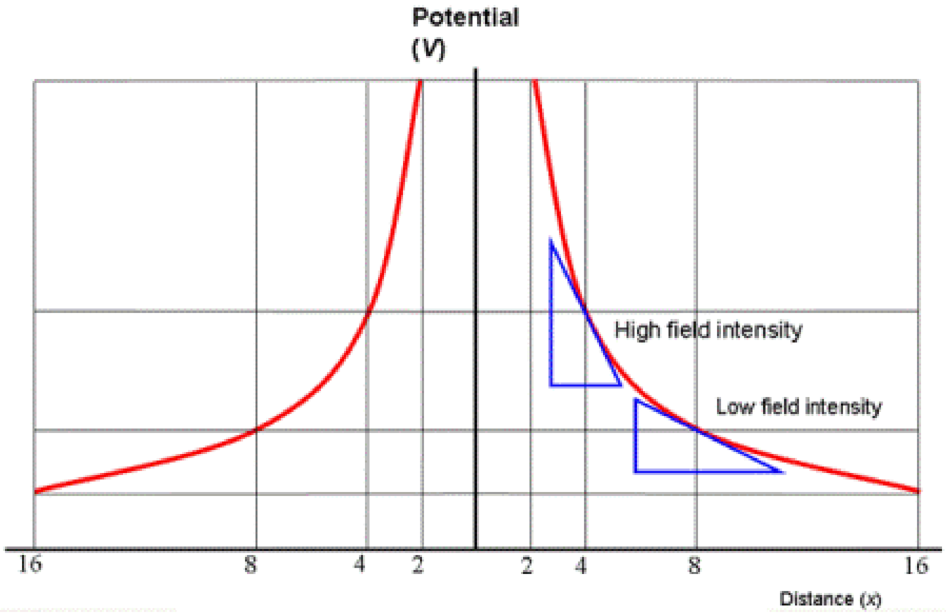
- Small increment: $dV = -\vec{E}.d\vec{l} = -E_x dx - E_y dy - E_z dz$

- Using partial differentials, $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

- Hence: $\vec{\boldsymbol{E}} = (\boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z) = \left(-\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}, -\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{y}}, -\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{z}}\right) = -\vec{\boldsymbol{\nabla}}\boldsymbol{V}$. The scalar $V$ carries the same information as the vector $\vec{E}$ !

# Electrostatic potential

- The electric field is the **vector gradient of potential**, $\vec{E} = -\vec{\nabla}V$
- Electric field lines point from high to low potential




# Electrostatic potential

- We can add **any constant** $c$ to the potential $V$ without changing the electric field, since $\vec{E} = -\vec{\nabla}(V + c) = -\vec{\nabla}V - \vec{\nabla}c$ where $\vec{\nabla}c = \vec{0}$

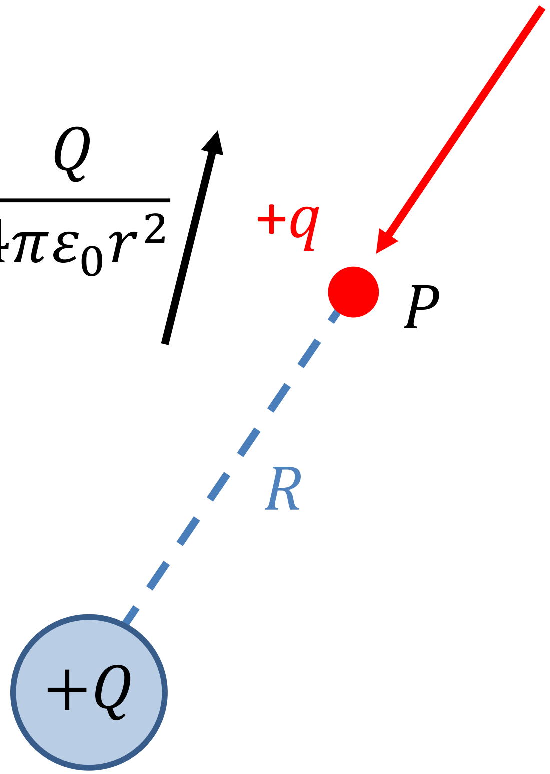
- **Only potential differences are meaningful**, the zero-point is arbitrary and we choose it at infinity for convenience

- The same is true in mechanics, we don't need to know the radius of the Earth to determine the speed of the roller coaster!

# Electrostatic potential

Example : potential at a point $P$ at radius $R$ from a charge $+Q$

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



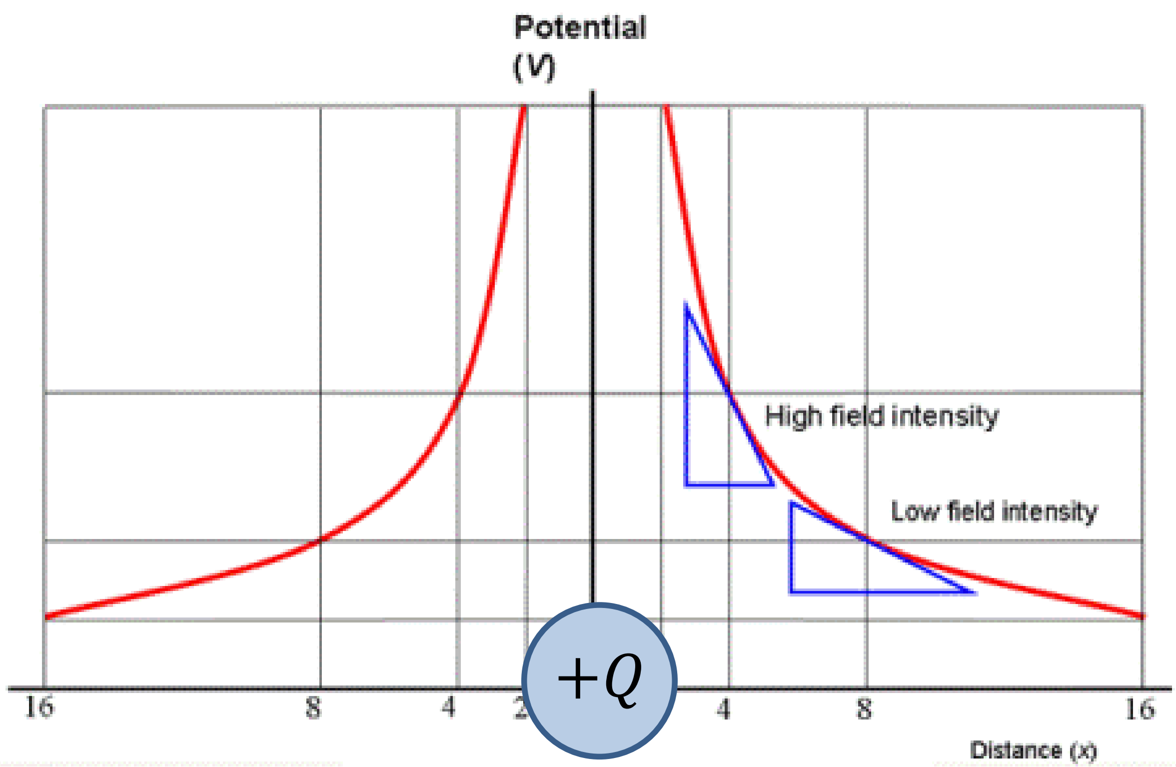

- Potential difference $\Delta V = -\int_A^B \vec{E}.d\vec{l}$
- Zero of potential is at infinity, and $\Delta V$ is independent of the path taken, so choose a radial path
- $V_P = -\int_\infty^R E\,dr = -\int_\infty^R \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$
- Solving the integral, $\boldsymbol{V_P = \frac{Q}{4\pi\varepsilon_0 R}}$

Please note in workbook

# Electrostatic potential

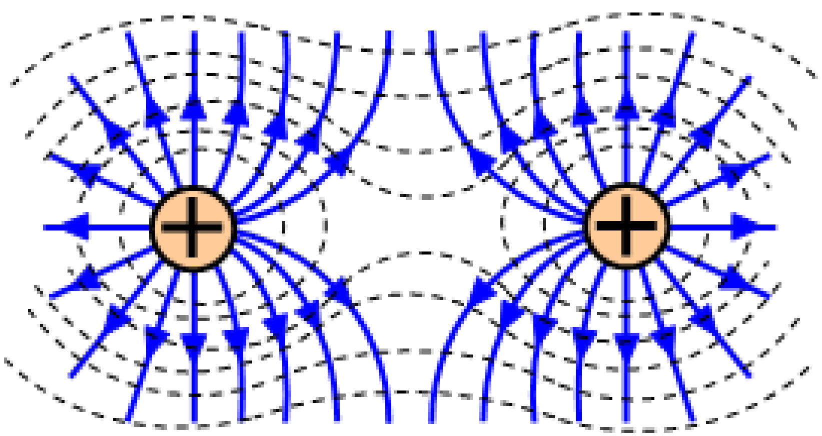
Example : potential at a point $P$ at radius $R$ from a charge $+Q$

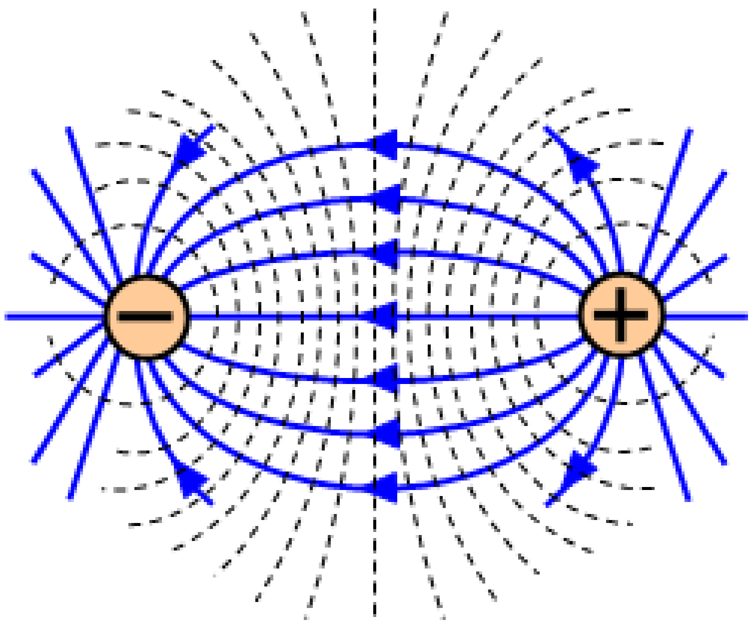
- The potential drops off as $1/R$, the electric field as $1/R^2$



# Electrostatic potential

- **Equipotentials** are lines of constant potential that run *perpendicular to electric field lines* (since $\vec{E} = -\vec{\nabla}V$).

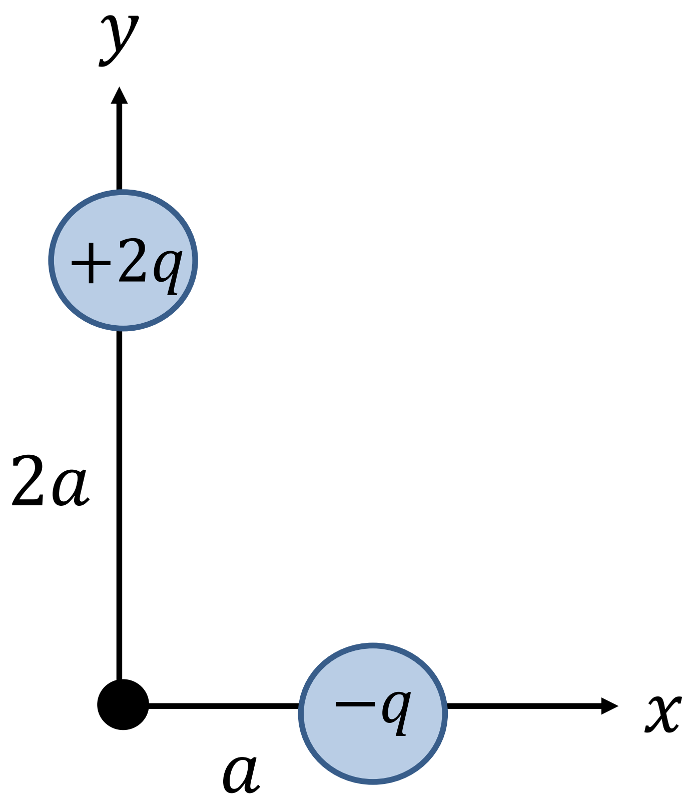




- A charge moving along an equipotential experiences no force

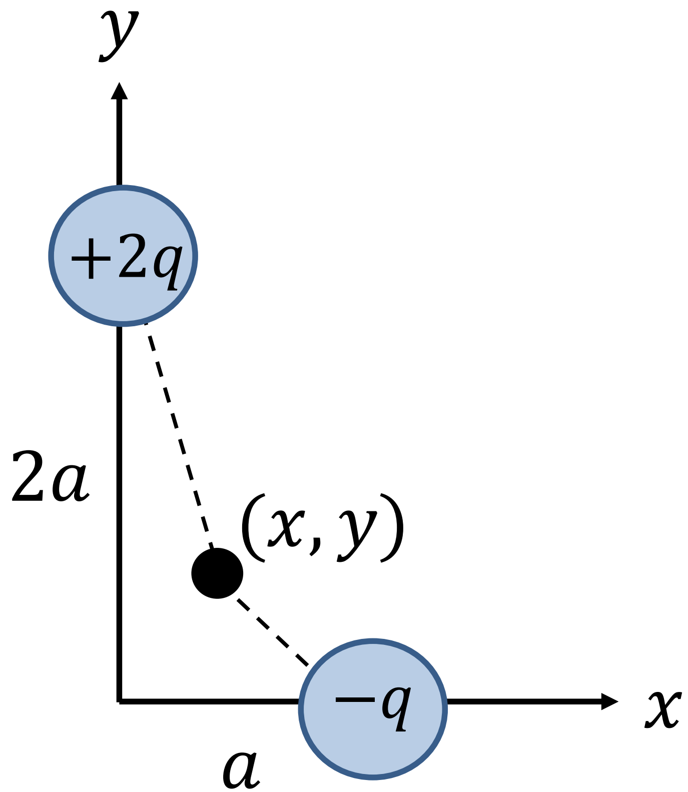
# Principle of superposition

- What is the electrostatic potential at the origin?



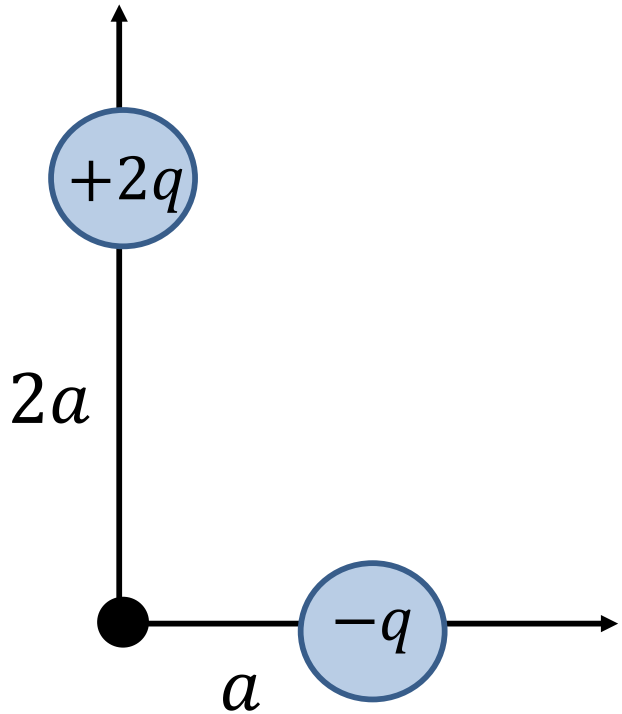
# Principle of superposition

- What is the electrostatic potential at $(x, y)$?



# Principle of superposition

- Using your expression for $V(x, y)$, what is the electric field at the origin, $\vec{E} = -\vec{\nabla}V$?

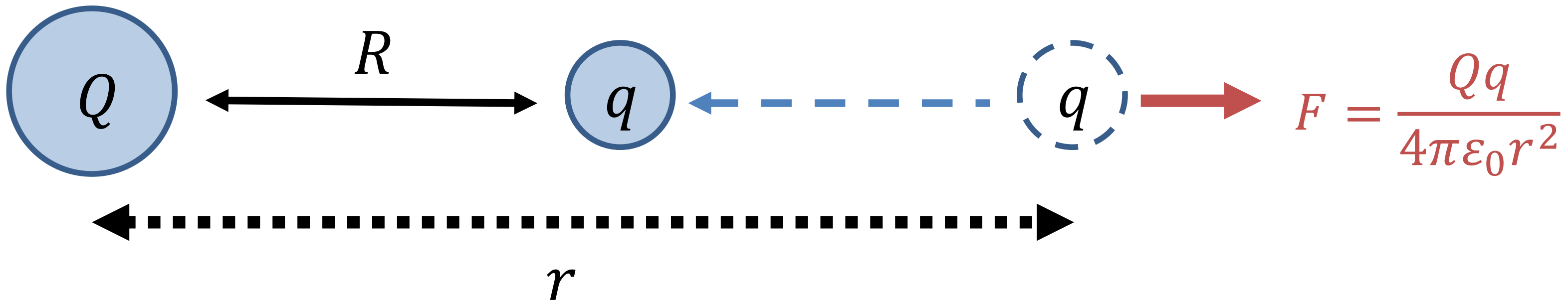


# Potential energy

- The **potential energy** of a system is the energy associated with its configuration in space
- It is equal to the **work done in assembling the system**

# Potential energy

- What work is needed to bring a charge $q$ to a distance $R$ from a charge $Q$?



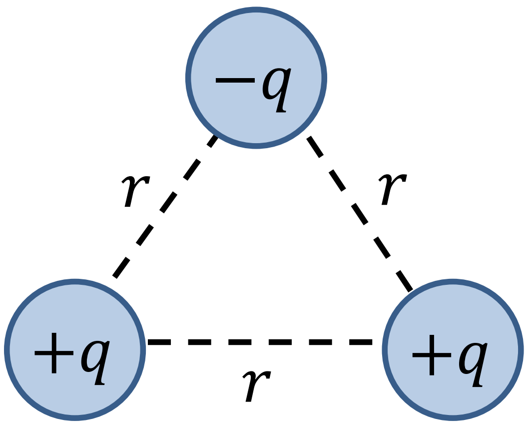

- This requires work $W = \int_{\infty}^{R} F\, dr = -\int_{\infty}^{R} \frac{Qq}{4\pi\varepsilon_0 r^2} = \frac{Qq}{4\pi\varepsilon_0 R}$

- This is the same as the **potential energy** $U$, which may be written $U = q\,V$ where $V$ is the electric potential at $R$

- For a *system of charges* $q_i$, $U = \frac{1}{2}\sum_{ij} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} = \frac{1}{2}\sum_i q_i V_i$

# Clicker question

What is the potential energy of this system of charges?

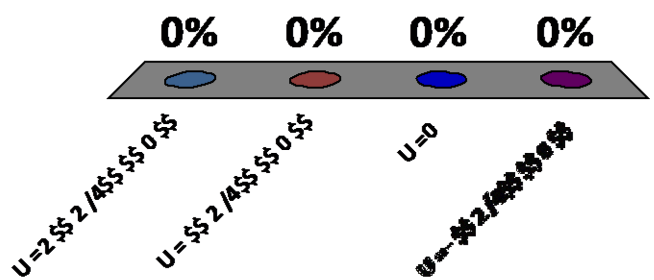



A. $U = 2q^2/4\pi\varepsilon_0 r$

B. $U = q^2/4\pi\varepsilon_0 r$

C. $U = 0$

D. $U = -q^2/4\pi\varepsilon_0 r$



# Summary

- The **electrostatic potential difference** $\Delta \boldsymbol{V}$ is the work done in moving a unit charge between two points, $\Delta V = -\int \vec{E}.d\vec{l}$. The zero-point is arbitrary and usually taken at infinity.

- The electrostatic potential $V$ carries all the information on the electric field, $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\boldsymbol{V}$, using the **gradient operator** $\vec{\nabla}$

- The **electrostatic potential energy** is the work done in assembling the system of charges

- A system of charges $q_i$ has **potential energy** $\frac{1}{2}\sum_i q_i V_i$