Class 2: Index Notation

In this class we will start developing index notation, the key mathematical basis of Relativity. We will also learn how to describe flows of energy and momentum.

Class 2: Index Notation

At the end of this session you should …

- … **know some examples of 4-vectors and tensors**, objects whose components transform between different inertial reference frames using the Lorentz transformations

- … **be developing some familiarity with index notation**: the difference between up- and down-indices, how one may be converted into the other, and summation rules

- … **understand how the density/flow of energy/momentum** may be described by the matter-energy tensor $T^{\mu\nu}$

What is a 4-vector?

- A **4-vector** is an array of 4 physical quantities whose values in different inertial frames are related by the Lorentz transformations

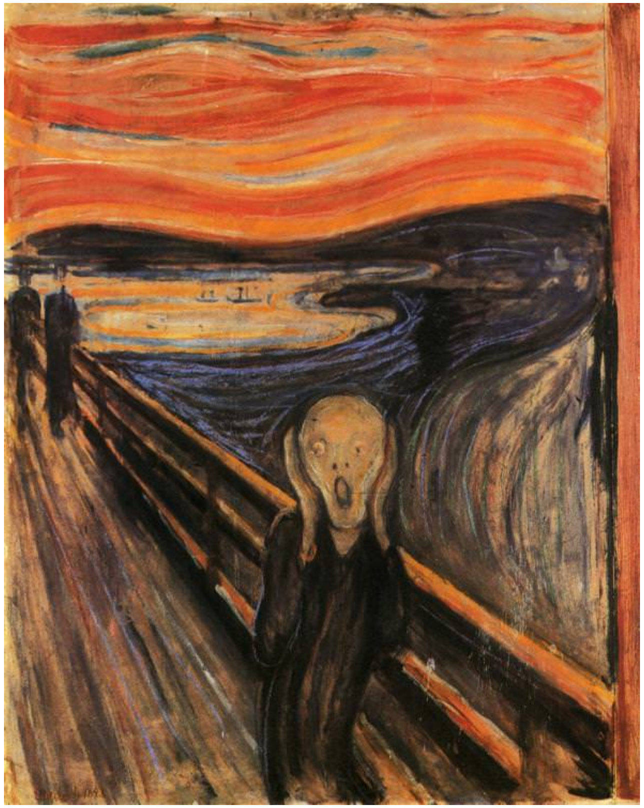
- The prototypical 4-vector is hence $\boldsymbol{x^{\mu} = (ct, x, y, z)}$

- Note that the index μ is a superscript, and can take four values $\mu = \{0,1,2,3\}$, one for each element (e.g., $x^0 = ct$). *It doesn't mean "to the power of".*

- We will meet subscript indices shortly!

Index notation

- When writing x^μ to describe an array of quantities, we are using "index notation" – the convenient mathematical approach for calculations in Relativity



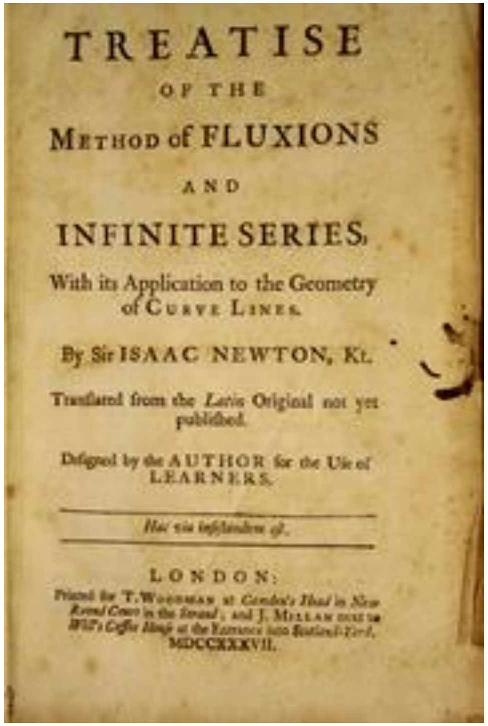
- For example, by the end of the unit we will be encountering equations like …

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\lambda\nu} - \partial_\nu \Gamma^\kappa_{\lambda\mu} + \Gamma^\kappa_{\mu\alpha}\Gamma^\alpha_{\lambda\nu} - \Gamma^\kappa_{\nu\alpha}\Gamma^\alpha_{\lambda\mu}$$

- *Aaargh!*

Index notation

- Good notation is always very important …



TREATISE
OF THE
METHOD of FLUXIONS
AND
INFINITE SERIES,
With its Application to the Geometry of CURVE LINES.
By Sir ISAAC NEWTON, Kt.
Translated from the *Latin* Original not yet published.
Designed by the AUTHOR for the Use of LEARNERS.

LONDON:
MDCCXXXVII.

https://archive.org/details/methodoffluxions00newt

- We will spend time practising using index notation

Producing 4-vectors

- **4-vectors are useful, because we know how their components transform between inertial frames**

- Since the Lorentz transformations are linear, the *sum/difference of 4-vectors* is also a 4-vector

- In particular, the difference in space-time co-ordinates is a 4-vector, $\boldsymbol{dx^{\mu} = (cdt, dx, dy, dz)}$

- New 4-vectors may also be obtained by *multiplying/dividing by an invariant*, such as the proper time interval $d\tau$ or the rest mass m_0

4-velocity and 4-momentum

- The 4-vector $v^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma u_x, \gamma u_y, \gamma u_z)$ is known as the **4-velocity of a particle** with 3D velocity $\vec{u} = (u_x, u_y, u_z)$

- The 4-vector $p^\mu = m_0 v^\mu = (\gamma m_0 c, \gamma m_0 u_x, \gamma m_0 u_y, \gamma m_0 u_z) = (\frac{E}{c}, p_x, p_y, p_z)$ is known as the **4-momentum of a particle**

- This immediately tells us how the **energy and momentum of a particle transform between frames**:

$$E' = \gamma(E - \frac{v p_x}{c}) \qquad p'_x = \gamma(p_x - \frac{vE}{c})$$

“Down” 4-vectors

- A **“down” 4-vector** in Special Relativity is obtained simply by reversing the sign of the first component of a 4-vector
- For example, a down 4-vector is $x_\mu = (-ct, x, y, z)$
- This will be a very useful device in calculations, as we will now explore!

Invariants in index notation

- We have seen that a useful quantity in Special Relativity is the space-time interval $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$
- In index notation, this can be written as $ds^2 = dx_0dx^0 + dx_1dx^1 + dx_2dx^2 + dx_3dx^3 = \sum_{\mu=0}^{3} dx_\mu dx^\mu$
- In index notation, this is abbreviated as $ds^2 = dx_\mu dx^\mu$
- **Greek indices which repeat on the top and bottom of an expression are always summed from 0 to 3**
- Note that *we can use any letter to indicate a summed index* – $dx_\nu dx^\nu$ and $dx_\alpha dx^\alpha$ are exactly the same!

Lorentz transformations

- The Lorentz transformations are written in index notation as $\boldsymbol{x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}}$
- This is "four equations in one", since $\mu = \{0,1,2,3\}$
- **Why??** The index ν appears on the top and bottom of the R.H.S. so is summed, leaving a single up-index μ

How can we make sense of "$L^{\mu}{}_{\nu} x^{\nu}$"??

"If in doubt, write it out ..."

Lorentz transformations

- Let's write it out explicitly:

$$x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu} = \sum_{\nu=0}^{3} L^{\mu}{}_{\nu} x^{\nu}$$

$$\mu = 0 \rightarrow \qquad x'^{0} = L^{0}{}_{0} x^{0} + L^{0}{}_{1} x^{1} + L^{0}{}_{2} x^{2} + L^{0}{}_{3} x^{3}$$

$$\mu = 1 \rightarrow \qquad x'^{1} = L^{1}{}_{0} x^{0} + L^{1}{}_{1} x^{1} + L^{1}{}_{2} x^{2} + L^{1}{}_{3} x^{3}$$

etc.

- It is analogous to a **matrix multiplication**:

$$\underbrace{\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}}_{x'^{\mu}} = \underbrace{\begin{pmatrix} \gamma & -v\gamma/c & 0 & 0 \\ -v\gamma/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L^{\mu}{}_{\nu}} \underbrace{\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}}_{x^{\nu}}$$

Raising and lowering an index

- The transformation from an "up" to a "down" 4-vector can be written as $\boldsymbol{x_\mu = \eta_{\mu\nu} x^\nu}$. Again, this is "four equations in one".
- $\eta_{\mu\nu}$ is a matrix $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ that reverses the 1st sign
- This is known as **lowering an index** $(x^\mu \rightarrow x_\mu)$
- Similarly, to **raise an index** we can write $x^\mu = \eta^{\mu\nu} x_\nu$, where $\eta^{\mu\nu}$ is the same matrix as above
- The same goes for 2D quantities, e.g. $L^{\mu\nu} = \eta^{\mu\lambda} L^\nu{}_\lambda$

Gradient transformations

- Consider a function of space-time co-ordinates $f(ct, x, y, z)$, which has gradients at a point $\left(\frac{1}{c}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. What are its gradients with respect to co-ordinates in S', (ct', x', y', z')?

- By the **chain rule**: $\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial f}{\partial x^{\nu}}$

- Since $x^{\nu} = L_{\mu}{}^{\nu} x'^{\mu}$, we have $\frac{\partial x^{\nu}}{\partial x'^{\mu}} = L_{\mu}{}^{\nu}$ ("if in doubt, write it out") so $\frac{\partial f}{\partial x'^{\mu}} = L_{\mu}{}^{\nu} \frac{\partial f}{\partial x^{\nu}}$

- The gradient of a function transforms using the Lorentz transformations: $\boldsymbol{\partial_{\mu} f} = \left(\frac{\mathbf{1}}{\boldsymbol{c}}\frac{\boldsymbol{\partial f}}{\boldsymbol{\partial t}}, \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial x}}, \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial y}}, \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial z}}\right)$ **is a down 4-vector**

Matter and energy

- To develop General Relativity we need to describe *how matter-energy is distributed*, and *where it's going*
- This is achieved by an object known as the **energy-momentum tensor** $\boldsymbol{T^{\mu\nu}}$ (at each point of space-time)
- For now, we can think of a "tensor" as a 2D matrix
- $T^{\mu\nu}$ has two indices because momentum has a direction, but can also be transported in different directions (e.g., a flux of x-momentum in the y-direction, if x-moving particles are drifting in y)

Matter and energy

- *It raises an immediate question: how does a quantity with 2 indices transform between different inertial reference frames S and S'?*

- The Lorentz transformation of a 4-vector x^{μ}:

$$x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$$

- The Lorentz transformation of a 2D tensor $T^{\mu\nu}$:

$$T'^{\mu\nu} = L^{\mu}{}_{\kappa}\, L^{\nu}{}_{\lambda}\, T^{\kappa\lambda}$$

Energy-momentum tensor

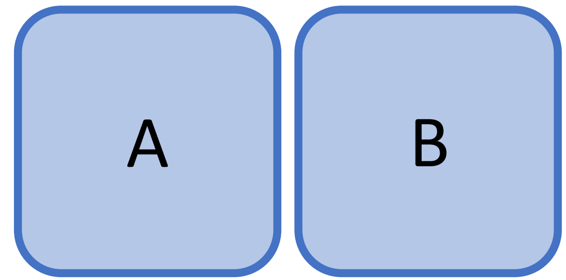
- Draw a box around a point in space-time containing a bunch of particles carrying energy and momentum
- If the box contains 4-momentum dp^μ and is moving with velocity $\frac{dx^\nu}{dt}$, we define $\boldsymbol{T^{\mu\nu} = \frac{dp^\mu}{dV}\frac{dx^\nu}{dt}}$
- Note that $T^{\mu\nu}$ is a "Lorentz-transforming quantity" because it is a product of two 4-vectors and a Lorentz scalar (the space-time volume element $dV\ dt$)
- *What are the different components of $T^{\mu\nu}$?*

Energy density and flow

- T^{00} is the **energy density** at a point
- $T^{0i} = T^{i0}$ $(i = 1,2,3)$ is the **flux of energy in the $\boldsymbol{i}$-direction** or the **$\boldsymbol{i}$-momentum density** $(\times \boldsymbol{c})$
- $T^{ij} = T^{ji}$ is the **flux of $\boldsymbol{i}$-momentum in the $\boldsymbol{j}$-direction** or the **flux of $\boldsymbol{j}$-momentum in the $\boldsymbol{i}$-direction**
- Hence the tensor is symmetric, $T^{\mu\nu} = T^{\nu\mu}$
- Let's get a better sense of what T^{ij} means ...

Energy density and flow

"The flux of i-momentum in the j-direction"? What does that mean??



- Consider two adjacent cubes of fluid A and B. In general A exerts a force $\vec{F}$ on B through the interface dS (and B exerts an equal-and-opposite force on A)

- $\vec{F}$ is equal to the rate at which momentum is pouring from A into B, such that the flux of momentum is $\vec{F}/dS$

- So T^{ij} is the **force per unit area** between adjacent elements

Perfect fluids

- Some forces, such as viscosity, act **parallel** to the interface between fluid elements

- For a **perfect fluid**, we only consider forces which act **perpendicular** to the interface, such that $T^{ii} =$ pressure P, and $T^{ij} = 0$

- For a non-relativistic perfect fluid,

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

- *This applies to the Universe!* (see later!)

Energy conservation

- We can express **energy-momentum conservation** using the relation

$$\partial_\mu T^{\mu\nu} = 0$$

- This is four equations in one again – 1 for energy and 3 for momentum

- It's a local relation which applies at every point of space-time