Class 2 : Computing the electric field

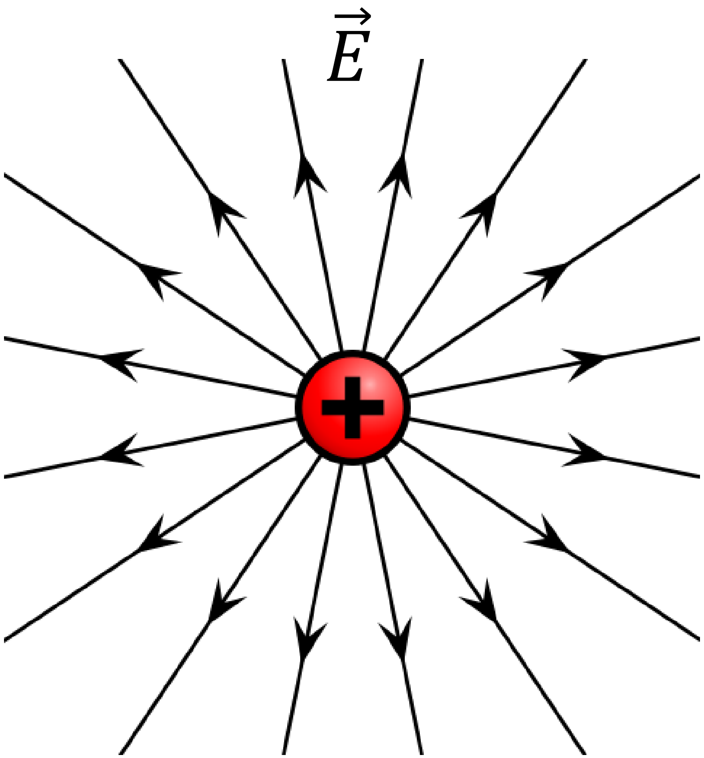
- Introducing Gauss's Law
- Concept of "flux over a surface"
- Symmetry arguments
- Example of applying Gauss's Law

Recap

- Electric charge is a fundamental property of nature; forces between charges are governed by **Coulomb's Law**

- We describe these effects by saying that an **electric field** $\vec{E}$ is set up in the region of space around charges

- A charge q placed in the electric field will feel a force $\vec{F} = q\vec{E}$



Introducing Gauss's Law

- For general distributions of charges, it would be a nightmare to work out the electric field through the superposition principle $\vec{E} = \sum_i \frac{q_i}{4\pi\varepsilon_0 {r_i}^2}\vec{\hat{r_i}}$

- Luckily we can use a more powerful method, **Gauss's Law**. This is equivalent to Coulomb's Law, but more convenient

- Gauss's Law says that ***for any closed surface S, the flux of the electric field $\vec{\mathbf{E}}$ through S is equal to the total charge enclosed by S, divided by $\boldsymbol{\varepsilon_0}$***

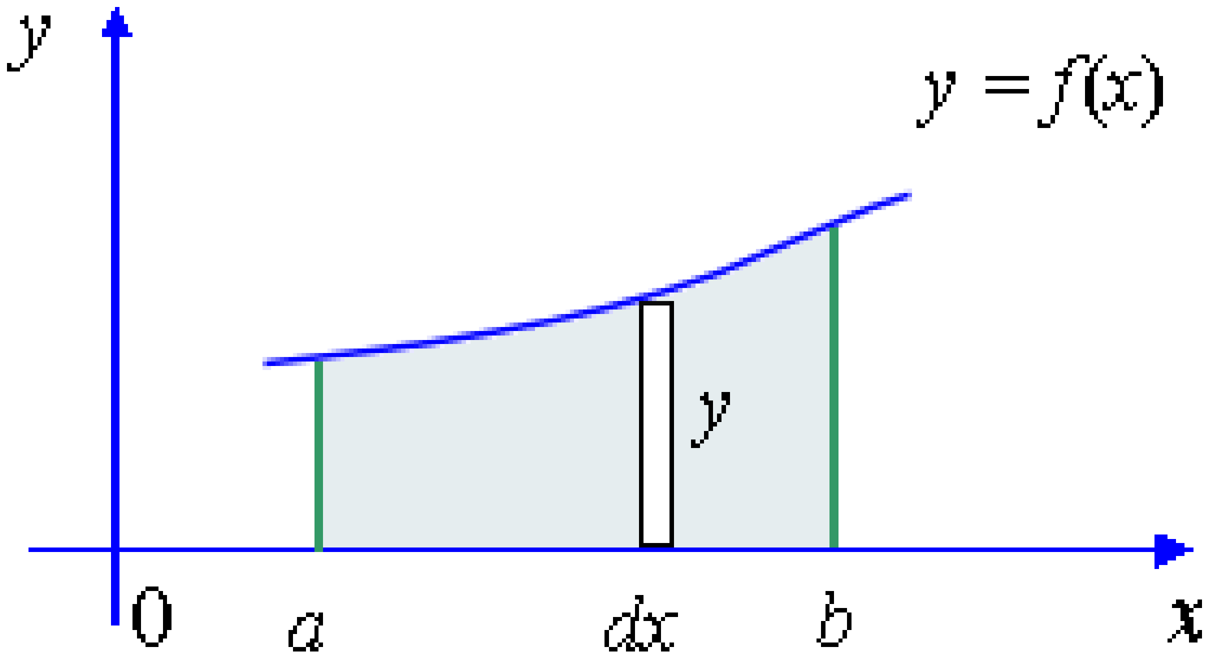
- In mathematics:

$$\int \vec{\boldsymbol{E}} \,.\, \boldsymbol{d}\vec{\boldsymbol{A}} = \frac{\boldsymbol{Q_{enclosed}}}{\boldsymbol{\varepsilon_0}}$$

Please note in workbook

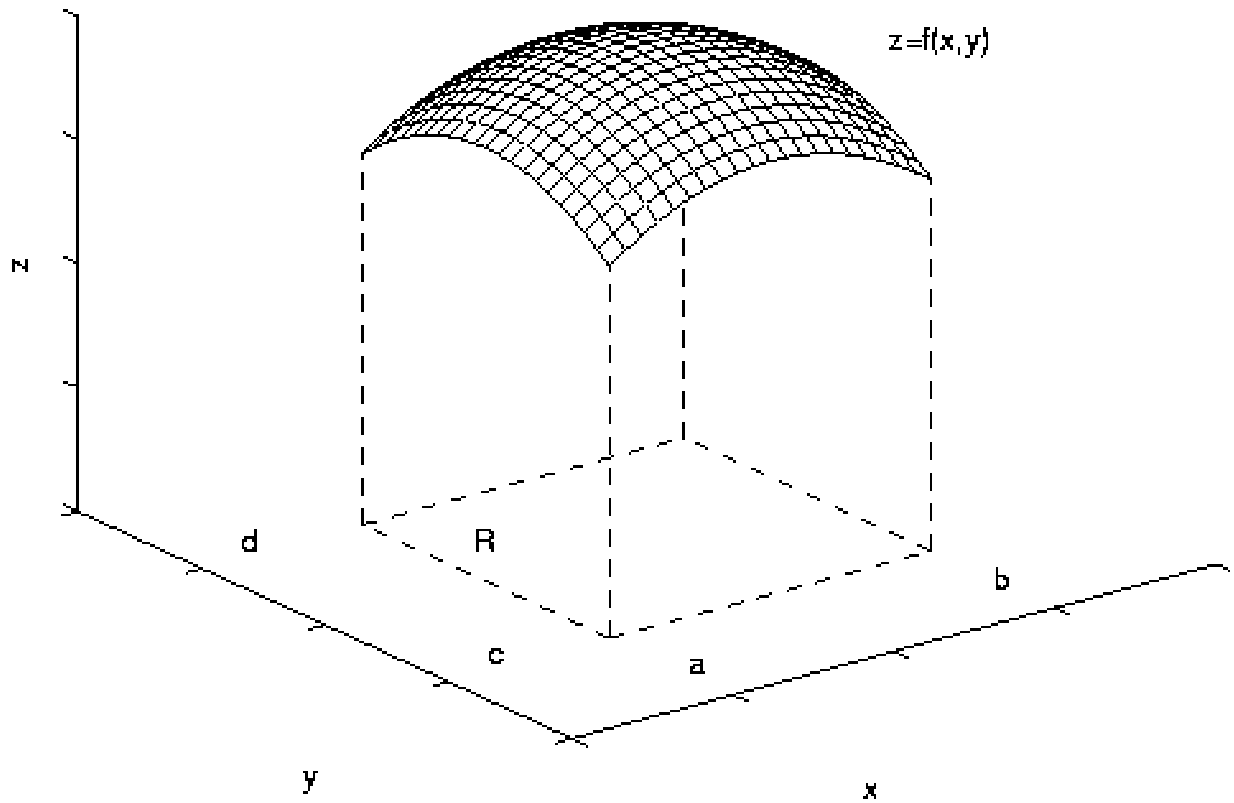
Surface integrals

- We are familiar with the concept of integration as summing up the “area under a curve” – written as $\int_a^b f(x)\,dx$



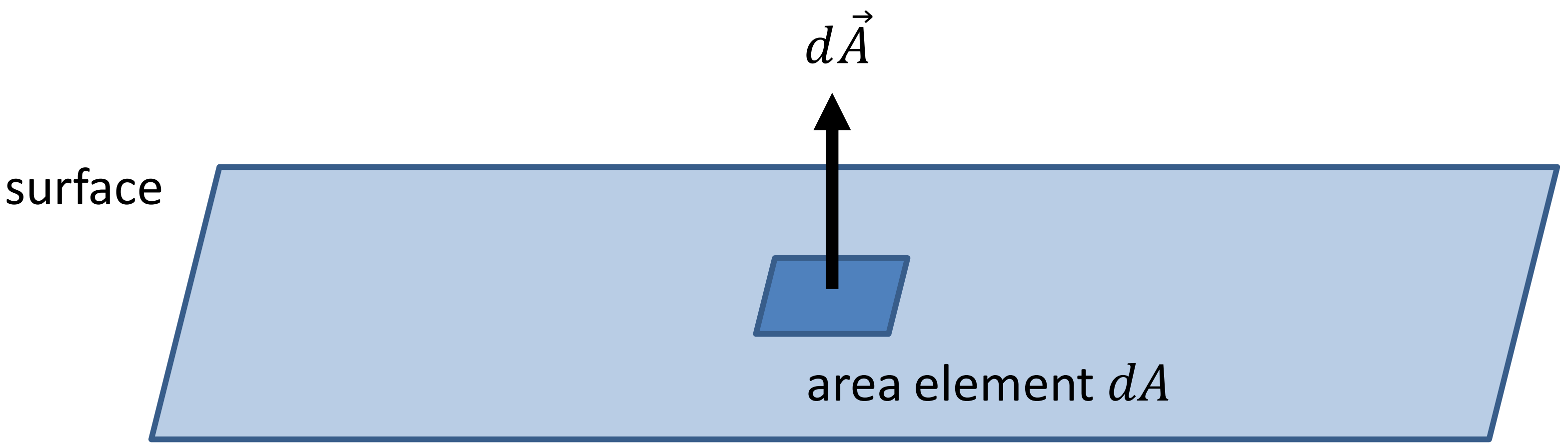
Surface integrals

- This can be generalized to more dimensions! A 2D integral sums up the volume under a surface, $\iint f(x, y)\, dx\, dy$



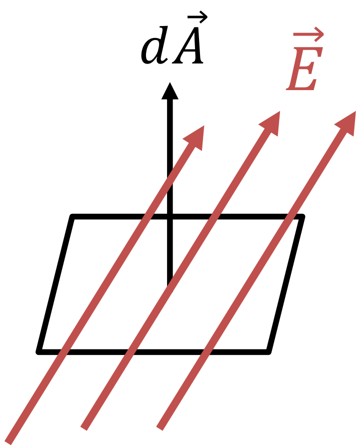
Surface integrals

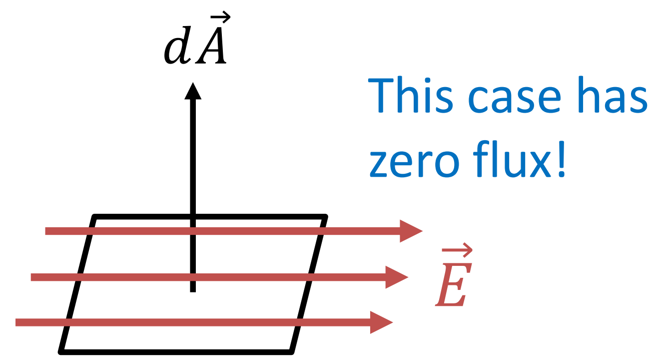
- The **flux over a surface** sums up the projection (dot product) of a vector field across the surface
- First, we **break the surface into area elements** $d\vec{A}$ (these are vectors, in the direction normal to the surface)



Surface integrals

- For each area element, we evaluate the dot product with the vector field, $\vec{E}.d\vec{A} = E\ dA\ \cos\theta$ (θ = angle between)



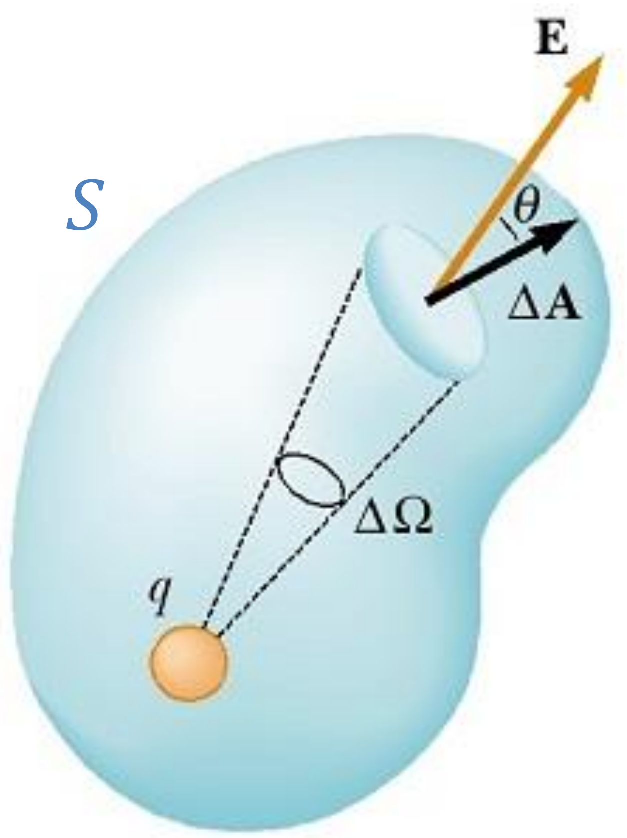


- We sum this up over the surface to obtain the **"flux"** $\int \vec{E}.d\vec{A}$

- **Important : we will only ever consider cases where $\vec{E}$ is perpendicular to the surface, so $\int \vec{E}.d\vec{A} = E \times Area$**

Introducing Gauss's Law

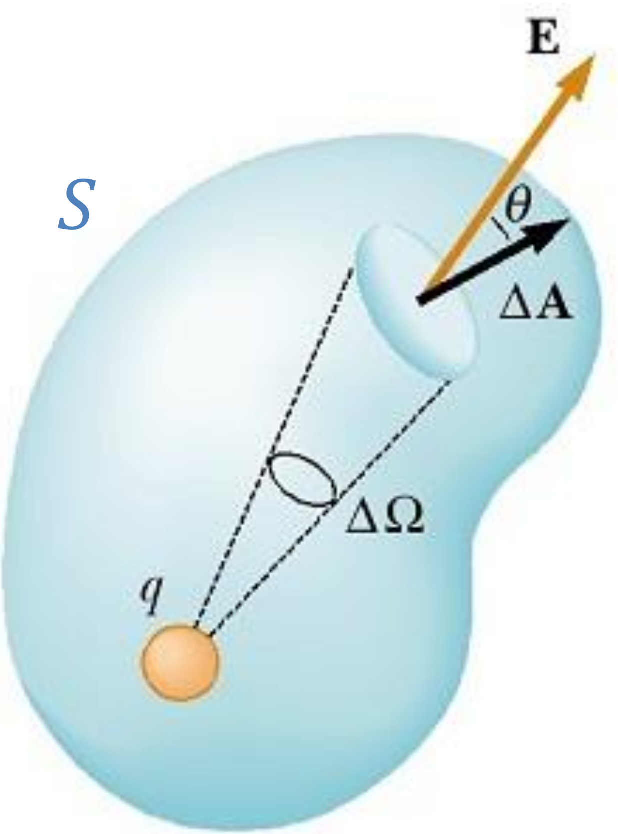
$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



- **Breaking it down …**
- S is any *closed* surface (i.e., with no edges/gaps)
- Consider a small area element $\Delta\vec{A}$ of S, where the vector is normal to S
- The flux of $\vec{E}$ through the area element is equal to $\vec{E}.\Delta\vec{A} = E\ \Delta A\ \cos\theta$
- The surface integral $\int \vec{E}.d\vec{A}$ means the total flux of $\vec{E}$ through the surface
- Gauss's Law says that this is equal to the total charge enclosed divided by ε_0

Introducing Gauss's Law

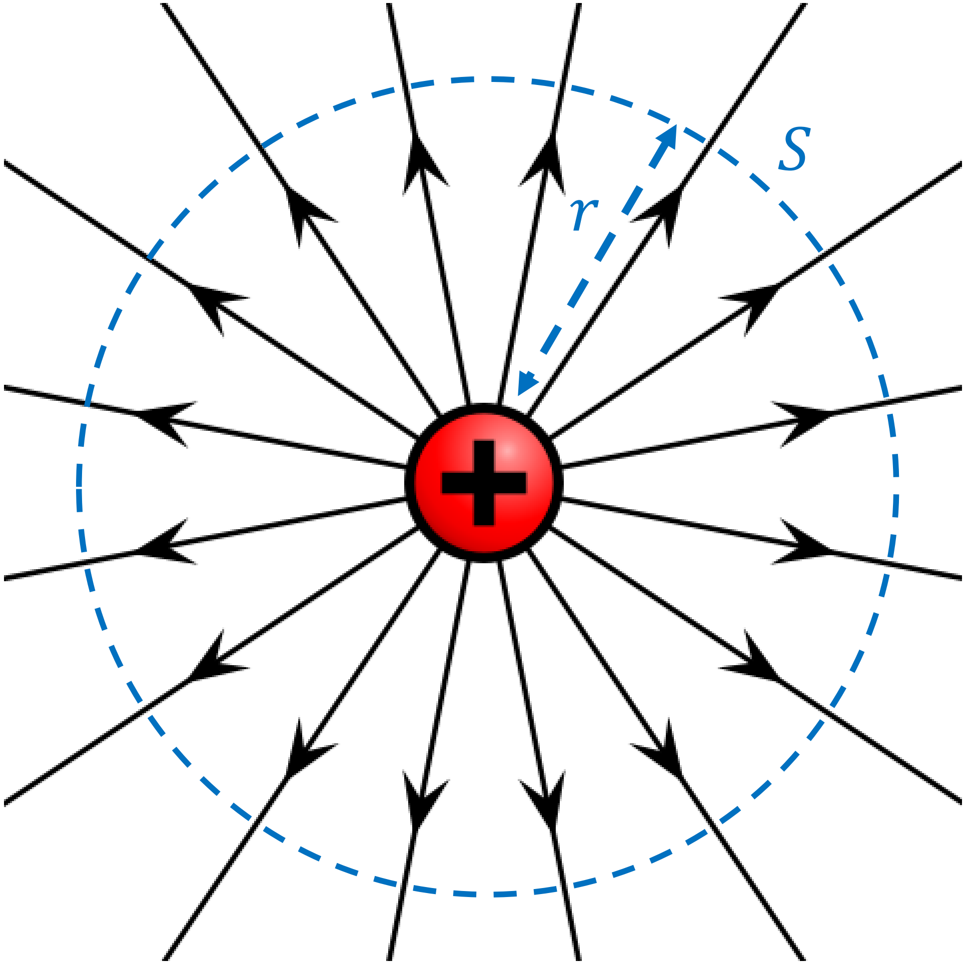
$$\int \vec{E} \,.\, d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



- **Why is it the same as Coulomb's Law? Let's derive Gauss's Law …**
- From the definition of solid angle we know that $\Delta\Omega = \frac{\Delta A \ \cos\theta}{r^2}$
- Coulomb's law : $E = \frac{q}{4\pi\varepsilon_0 r^2}$
- Flux through $\Delta\vec{A}$ is $\vec{E}.\Delta\vec{A} = E\ \Delta A\ \cos\theta = \frac{q\ \Delta A\ \cos\theta}{4\pi\varepsilon_0 r^2} = \frac{q\ \Delta\Omega}{4\pi\varepsilon_0}$
- Integrating over the whole surface: $\int \vec{E}.d\vec{A} = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{\varepsilon_0}$
- This holds true for any surface S

Introducing Gauss's Law

Example: point charge $+Q$

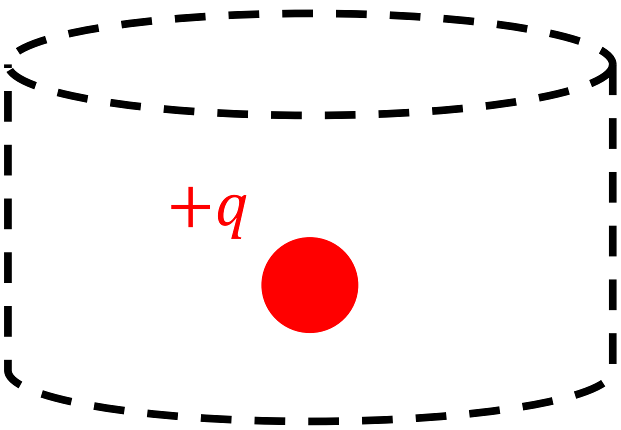


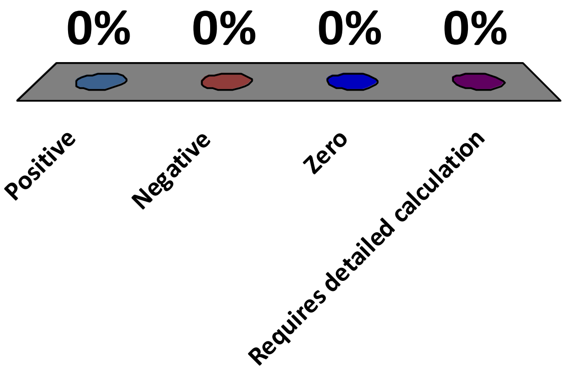
$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

- By spherical symmetry, the electric field $\vec{E}$ is radial
- Choose a spherical surface S of radius r centred on the charge
- $\vec{E}$ has the same magnitude across the surface and cuts it at right angles, so $\vec{E}.\,d\vec{A} = E \times A$
- Gauss's Law then simplifies to: $E \times 4\pi r^2 = \frac{Q}{\varepsilon_0}$
- Re-arranging: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

Clicker question

A positive point charge q is placed **inside** a closed cylinder S. What is the flux of $\vec{E}$ through S?





A. Positive

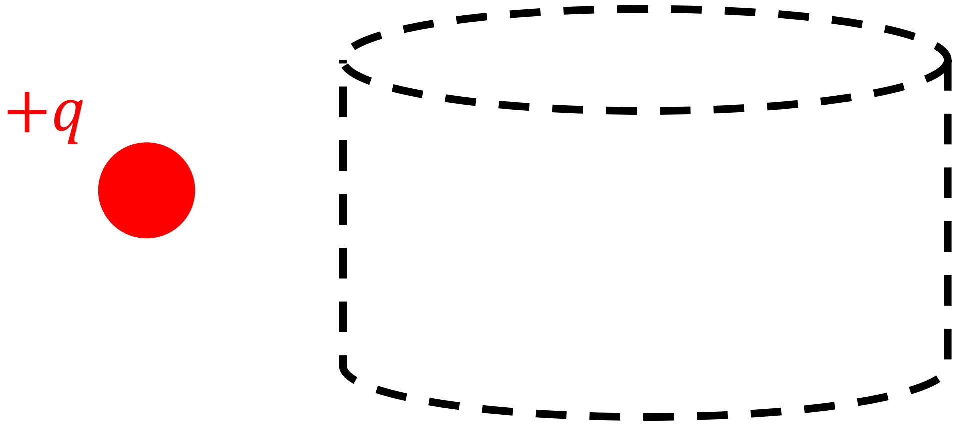
B. Negative

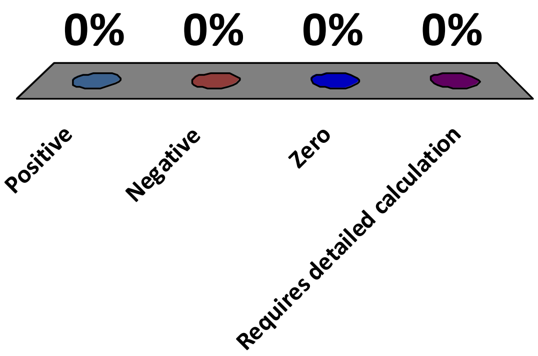
C. Zero

D. Requires detailed calculation

Clicker question

A positive point charge q is placed **outside** a closed cylinder S. What is the flux of $\vec{E}$ through S?





A. Positive

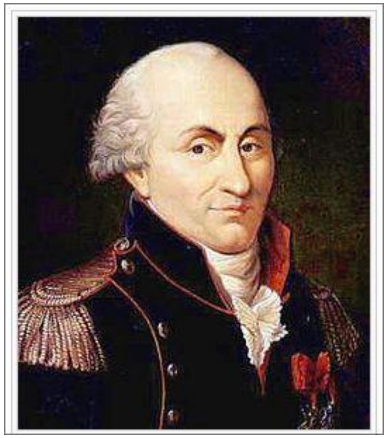
B. Negative

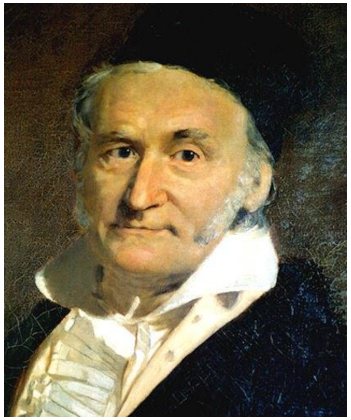
C. Zero

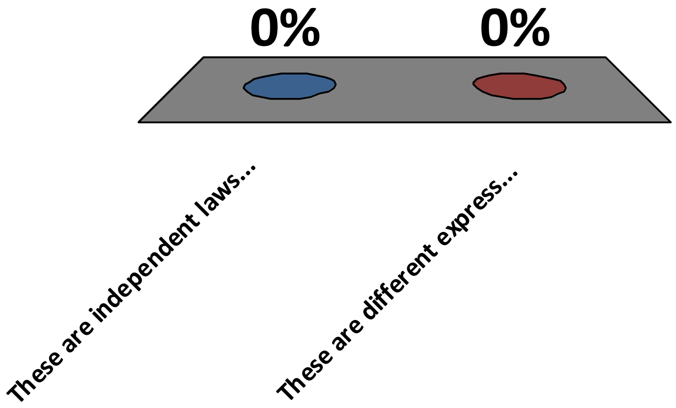
D. Requires detailed calculation

Clicker question

How would you best compare Coulomb's Law and Gauss's Law?



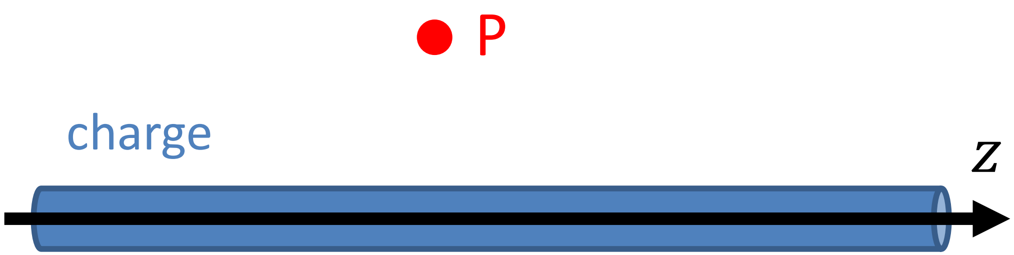


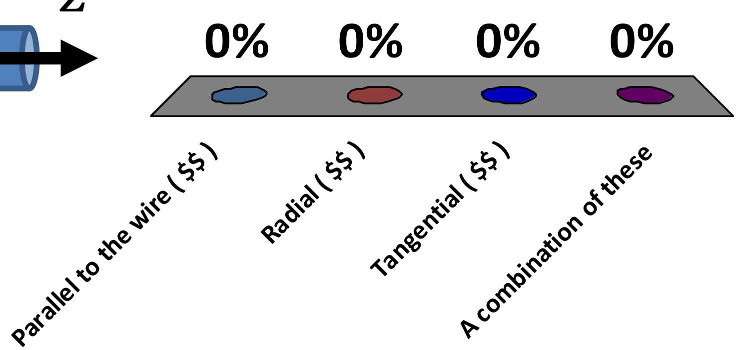


A. These are independent laws of physics applying in different situations
B. These are different expressions of the same law of physics

Clicker question

Consider a “infinite line of charge” along the z-axis of polar co-ordinates (r, θ, z). What is the direction of the electric field at point P?





A. Parallel to the wire ($\hat{z}$)

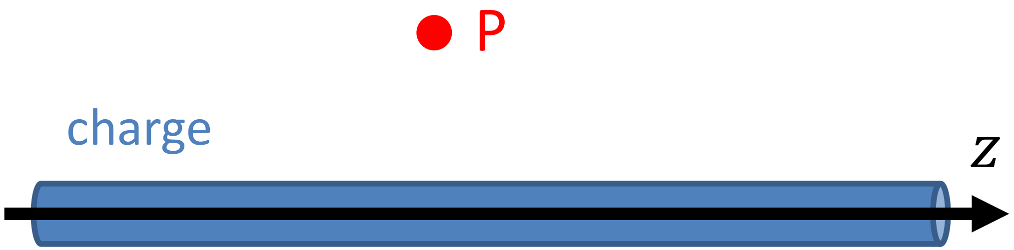
B. Radial ($\hat{r}$)

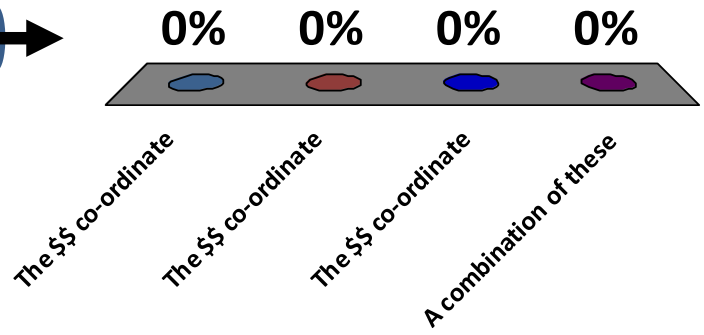
C. Tangential ($\hat{\theta}$)

D. A combination of these

Clicker question

Consider a "infinite line of charge" along the z-axis of polar co-ordinates (r, θ, z). On which of these variables does $|\vec{E}|$ at P depend?





A. The z co-ordinate

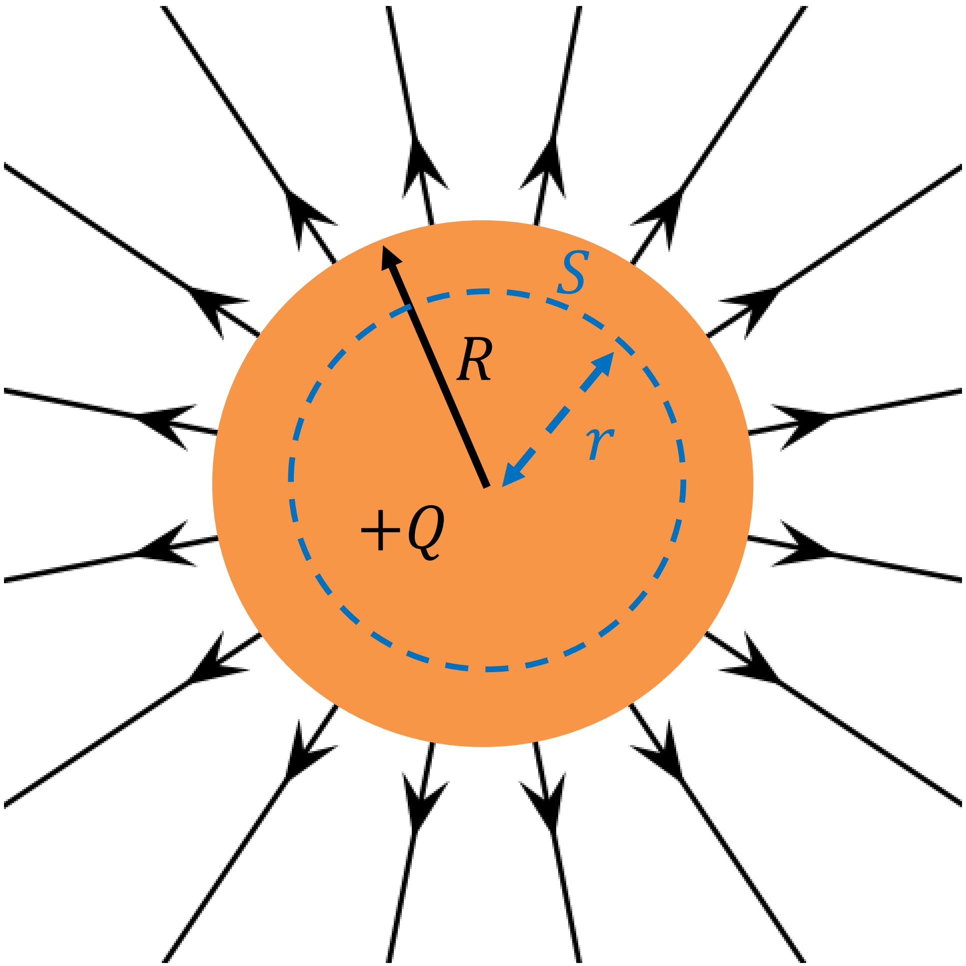
B. The r co-ordinate

C. The θ co-ordinate

D. A combination of these

Applying Gauss's Law

Example: uniform sphere of charge $+Q$ of radius R



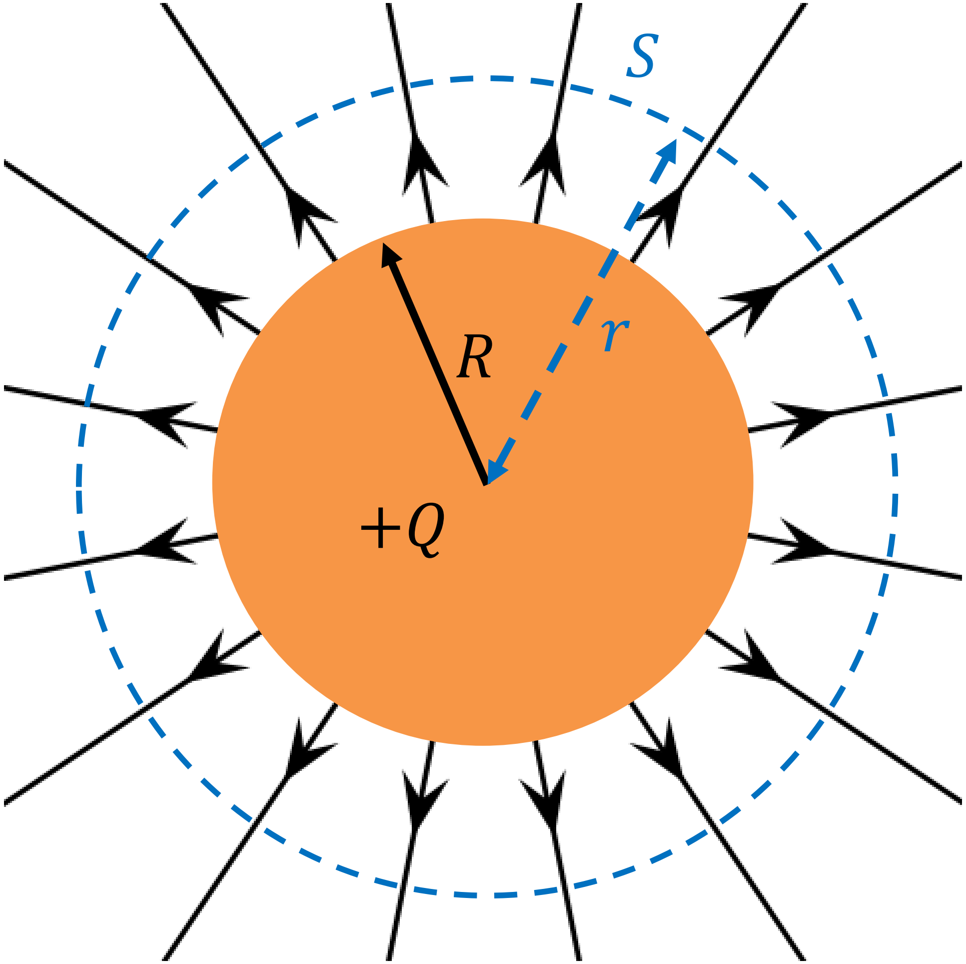
$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

- Adding up all the electrostatic forces would be a nightmare!
- By spherical symmetry $\vec{E}$ is radial so choose a spherical surface S of radius r as before
- **First case : if $r < R$**
- $Q_{enclosed} = Q \times \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = Q\,\frac{r^3}{R^3}$
- Gauss's Law: $E \times 4\pi r^2 = \frac{1}{\varepsilon_0} Q\,\frac{r^3}{R^3}$
- Hence $\boldsymbol{E = \frac{Q\,r}{4\pi\varepsilon_0 R^3}}$ **if $r < R$**

Applying Gauss's Law

Example: uniform sphere of charge $+Q$ of radius R

$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

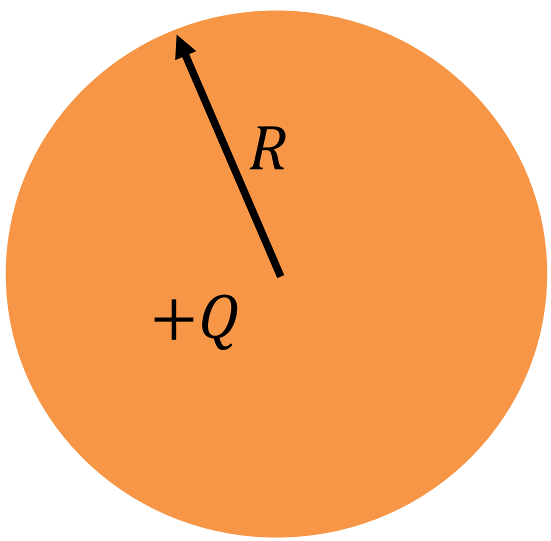


- **Second case : if $r > R$**
- $Q_{enclosed} = Q$
- Gauss's Law: $E \times 4\pi r^2 = \frac{Q}{\varepsilon_0}$
- Hence $\boldsymbol{E} = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\varepsilon_0 r^2}}$ **if $r > R$**
- Behaves like a point charge!

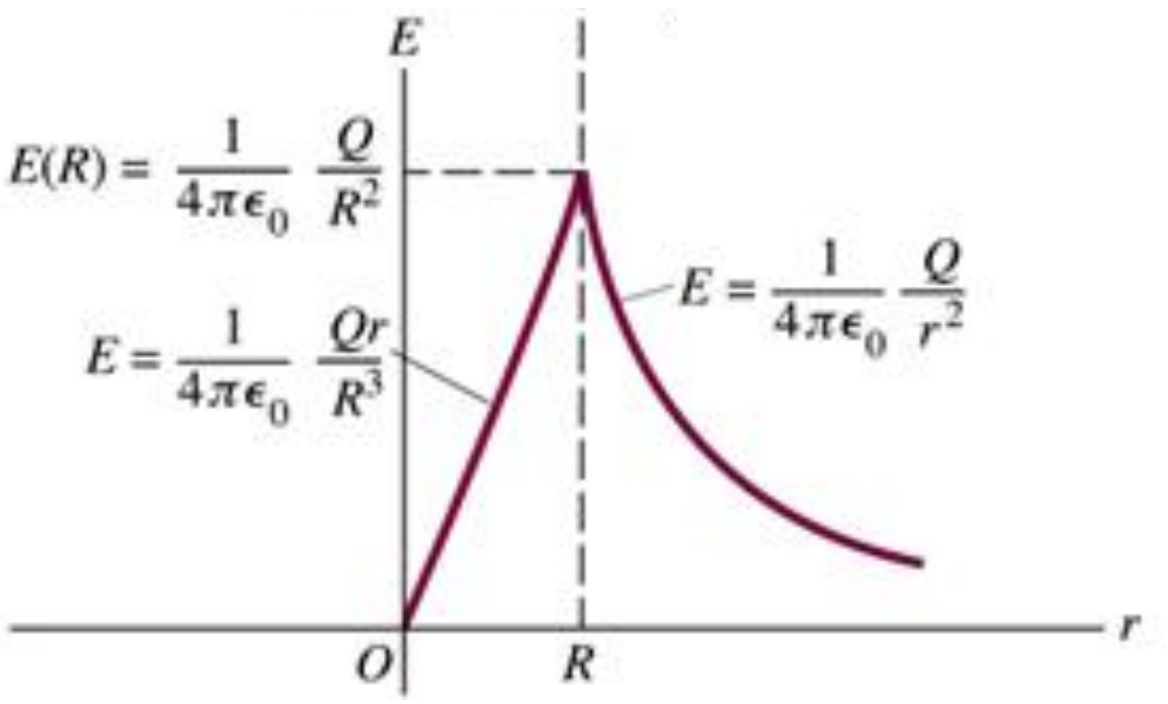
Applying Gauss's Law

Example: uniform sphere of charge $+Q$ of radius R

$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



Putting the two cases together:



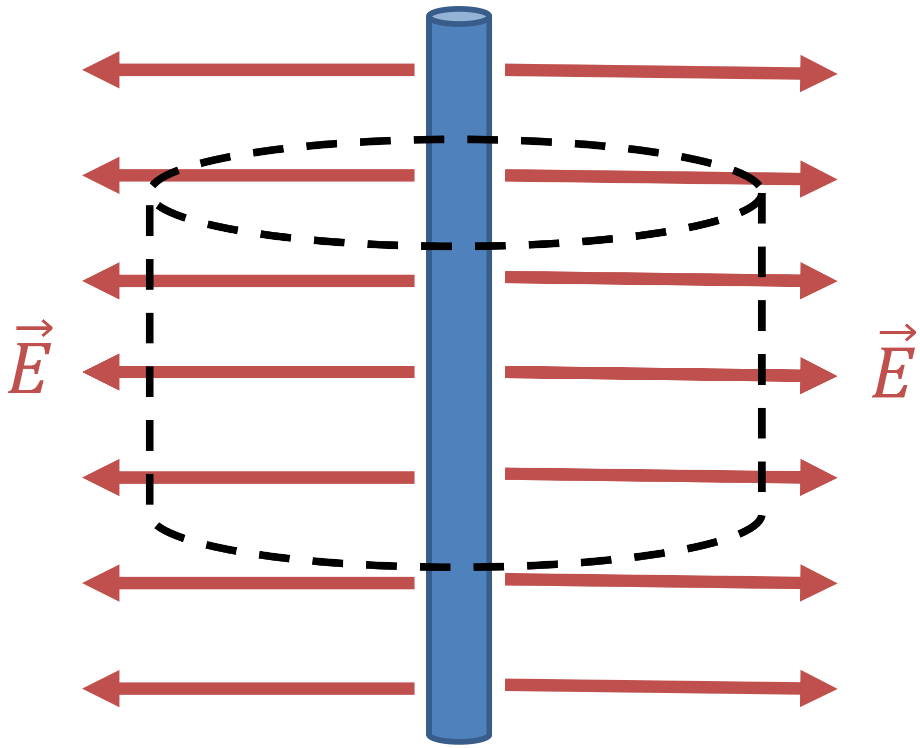
Applying Gauss's Law

- The same is true for gravity (which obeys the same inverse-square force law)
- Outside the Earth, its gravity is the same as if it were replaced by a point mass M_{Earth} at its centre

Applying Gauss's Law

Example: uniform charged wire (with charge λ C/m)

$$\int \vec{E} \, . \, d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

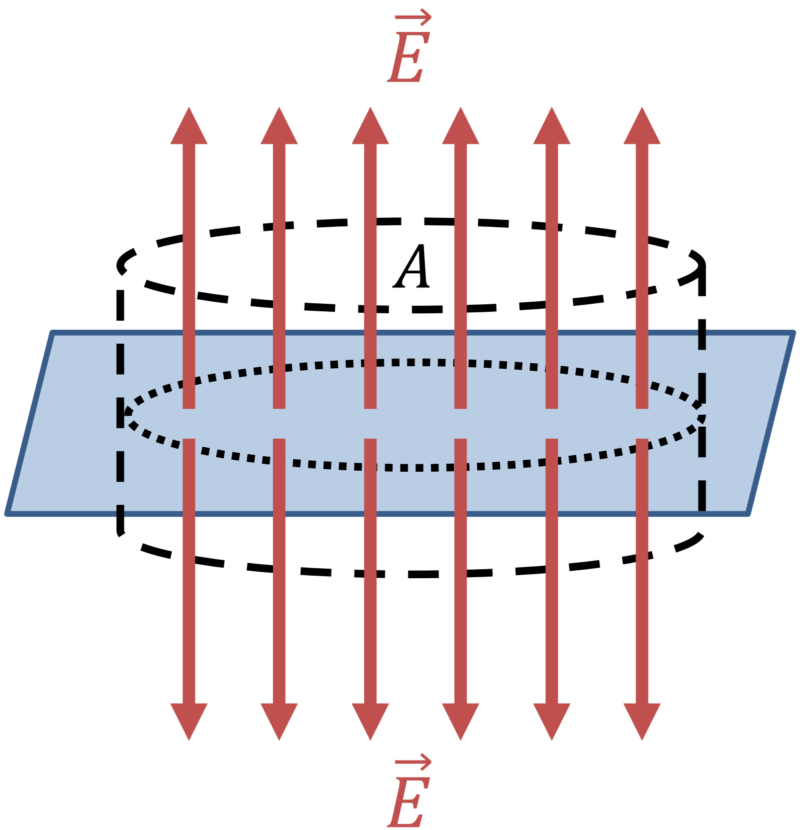


- By axial symmetry, the electric field $\vec{E}$ is radially outward from the wire
- Choose a cylindrical surface around the wire of radius r and height L
- On the flat edges, $\vec{E}$ is tangential to the surface, so $\vec{E} . d\vec{A} = 0$
- On the curved surfaces, $\vec{E}$ has the same magnitude and cuts at right angles, so $\int \vec{E} . d\vec{A} = E \times 2\pi r L$
- Charge enclosed $Q_{enclosed} = \lambda L$
- Gauss's Law: $E = \frac{\lambda}{2\pi\varepsilon_0 r}$

Applying Gauss's Law

Example: infinite sheet of charge (with charge $\sigma\ Cm^{-2}$)

$$\int \vec{E}\,.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$



- By symmetry, the electric field $\vec{E}$ is perpendicular to the sheet and symmetric about it
- Choose a cylindrical surface with area A crossing the sheet
- On the curved surfaces, $\vec{E}$ is tangential to the surface, so $\vec{E}.\,d\vec{A} = 0$
- On the flat edges, $\vec{E}$ cuts at right angles, so $\int \vec{E}.\,d\vec{A} = E \times 2A$
- Charge enclosed $Q_{enclosed} = \sigma A$
- Gauss's Law: $\boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}}$

Summary

- **Gauss's Law** is a powerful means of deducing the electric field $\vec{E}$ of a symmetric charge distribution

- Example symmetric charge distributions are spherical, a line or a sheet

- It is **equivalent to Coulomb's Law**, but much easier to apply in practice