Class 15 : Electromagnetic Waves

- Wave equations
- Why do electromagnetic waves arise?
- What are their properties?
- How do they transport energy from place to place?

Recap (1)

- In a region of space containing no free charge ($\rho_f = 0$) or current ($\vec{J}_f = \vec{0}$), Maxwell's equations can be written as

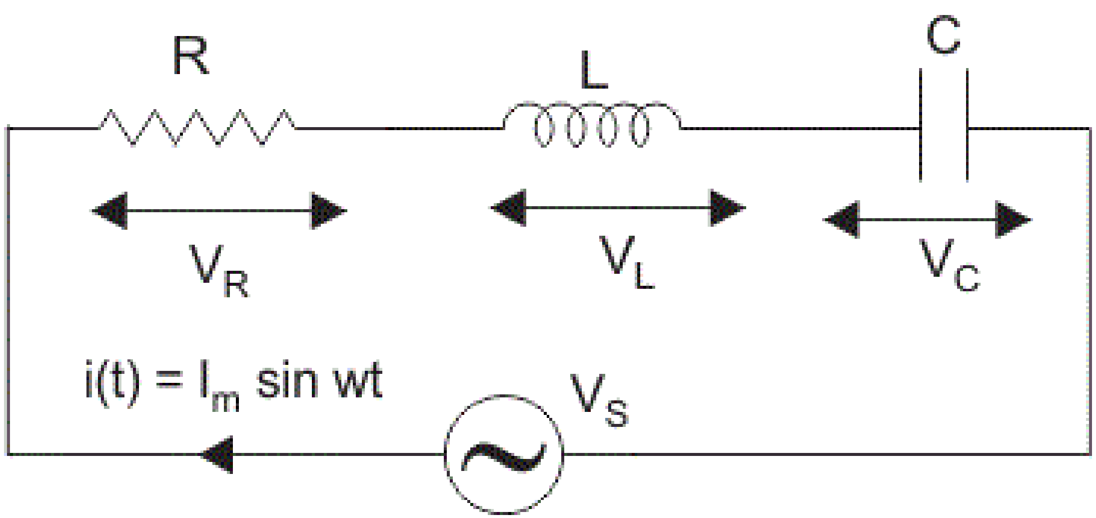
$$\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{E}} = \mathbf{0} \qquad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{B}} = \mathbf{0} \qquad \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \mu_0 \varepsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

Recap (2)

- **Work** needs to be performed when creating an $\vec{E}$-field (e.g. by assembling charges), or a $\vec{B}$-field (e.g. by establishing a current against an inductance)

- This creates a **potential energy stored in the electromagnetic field** with energy densities $\frac{1}{2}\varepsilon_0 E^2$ and $\frac{B^2}{2\mu_0}$, respectively



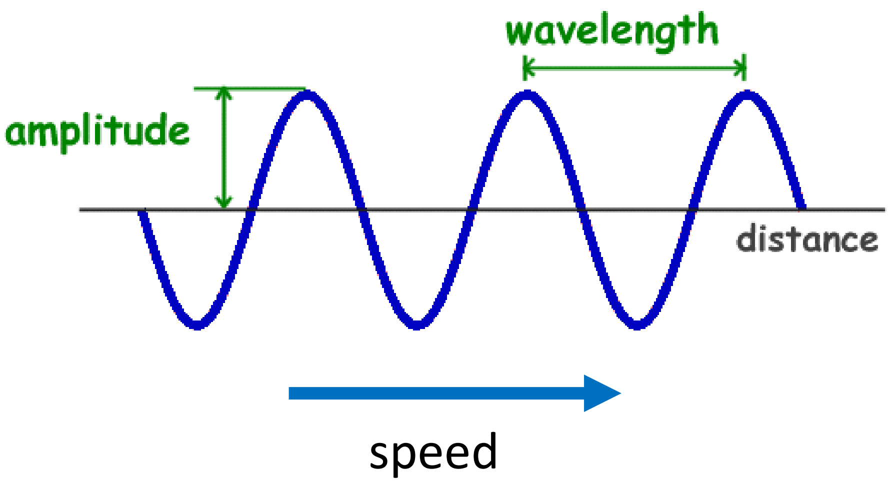
Wave equations

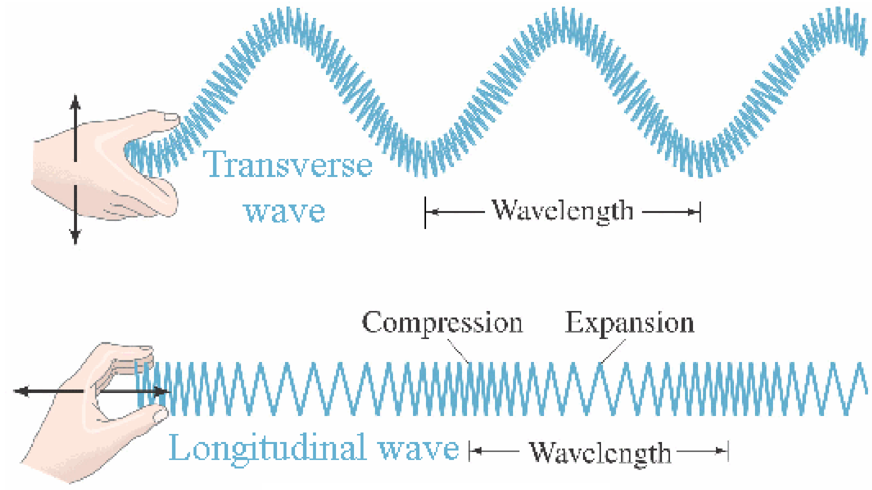
- We will now explore how these equations lead to **wave motion** called light – an *electromagnetic wave*

Wave equations

- A wave is a *propagating disturbance that carries energy from one point to another*

- A wave is characterized by its **amplitude**, **wavelength** and **speed** and can be *longitudinal* or *transverse*





Wave equations

- To discover EM waves, consider evaluating $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right)$

- Substituting in Maxwell's equations, this equals $\vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{dt}\left(\vec{\nabla} \times \vec{B}\right) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

- Also by vector calculus, $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - \vec{\nabla}^2 \vec{E}$

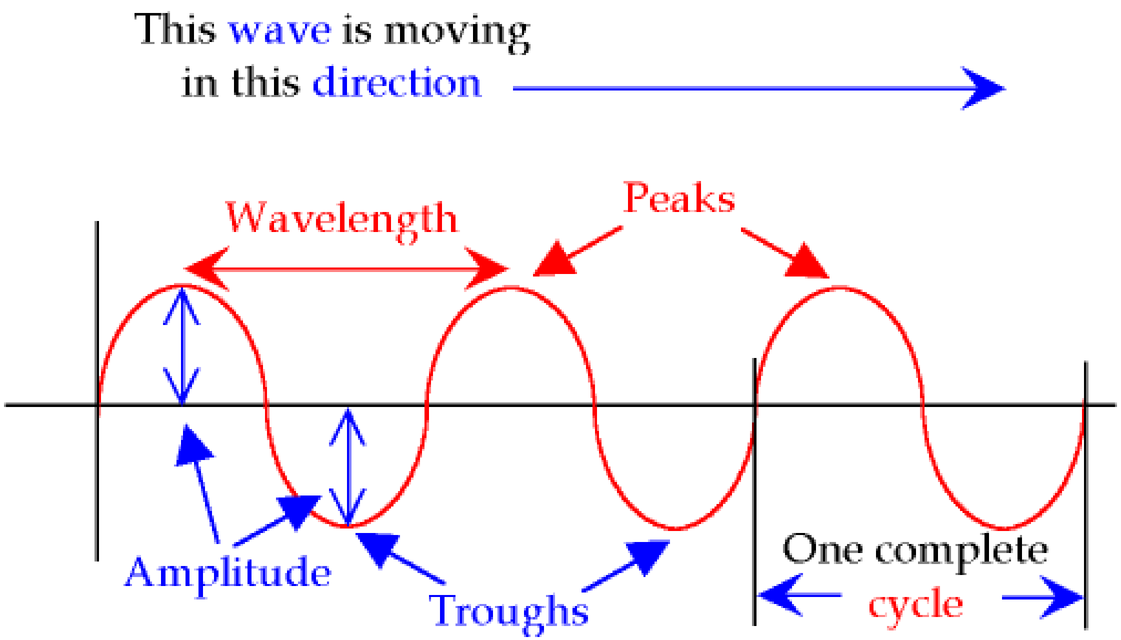
- Using Maxwell's equation $\vec{\nabla}.\vec{E} = 0$, this just equals $-\vec{\nabla}^2 \vec{E}$

- Putting the two sides together, we find $\boldsymbol{\vec{\nabla}^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$

Wave equations

- The equation represents a **travelling wave**

- To see why, consider a 1D wave travelling along the x-axis with displacement $y(x,t) = A\sin(kx - \omega t)$ in terms of its amplitude A, wavelength $\lambda = \frac{2\pi}{k}$, time period $T = \frac{2\pi}{\omega}$, velocity $v = \frac{\lambda}{T} = \frac{\omega}{k}$

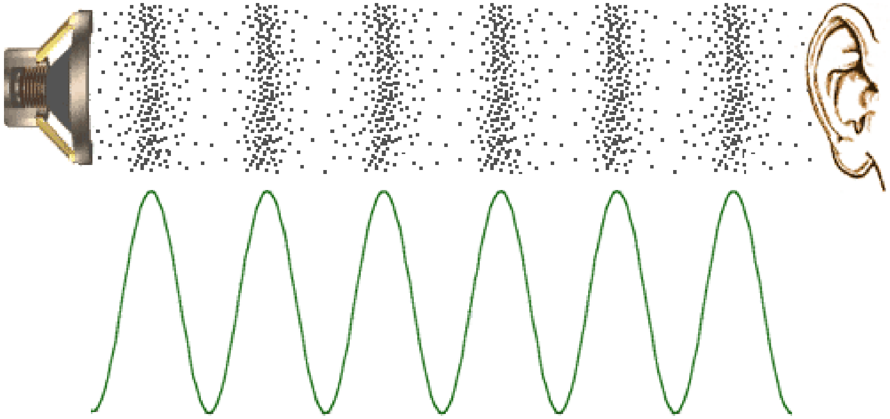


Wave equations

- **The displacement $\boldsymbol{y(x,t)}$ of a wave travelling with speed $\boldsymbol{v}$ satisfies the wave equation $\boldsymbol{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$**
- To check this, substitute in $y(x,t) = A\sin(kx - \omega t)$ where $v = \frac{\omega}{k}$
- On the left-hand side: $\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t)$
- On the right-hand side: $\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$
- Using $\frac{1}{v^2} = \frac{k^2}{\omega^2}$, we recover the wave equation!

Wave equations

- **The equation** $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ **describes general waves**



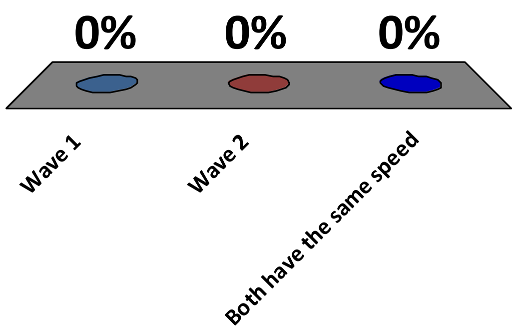
Clicker question

Two waves are described by the equations $y_1(x,t) = 2\sin(2x - t)$, $y_2(x,t) = 4\sin(x - 0.8t)$. Which wave has the higher speed?

A. Wave 1

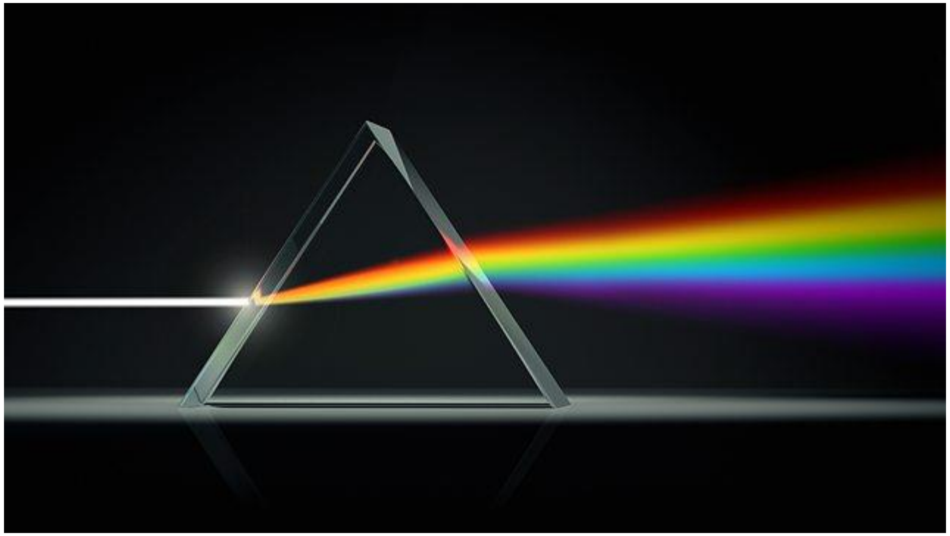
B. Wave 2

C. Both have the same speed



Wave equations

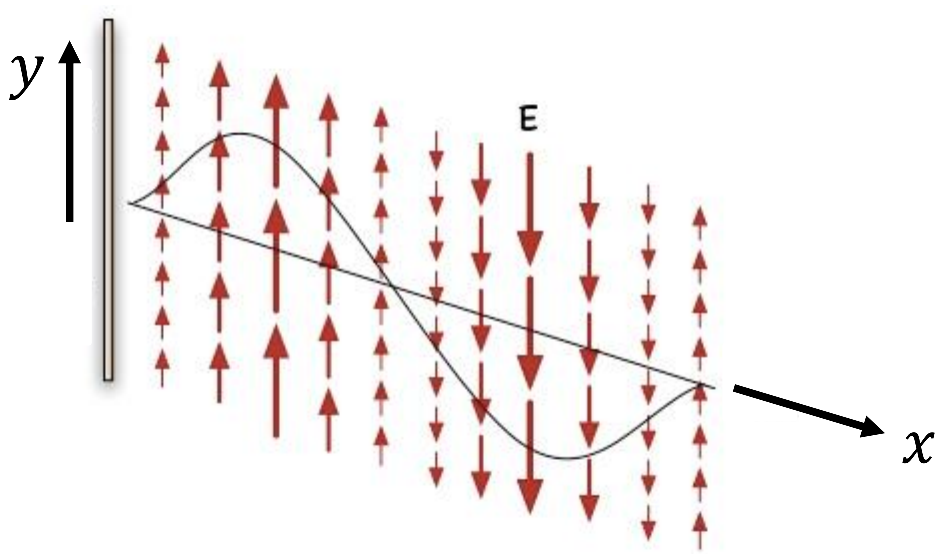
- The relation $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ implies that **each component of $\vec{E}$ is a wave travelling at speed $\boldsymbol{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}}$**
- We obtain $v = c = 3 \times 10^8$ m/s : the **speed of light**



Properties of the waves

- Let's find out more about the properties of electromagnetic waves

- To simplify the case, suppose that the $\vec{E}$-field is along the y-axis, such that $E_y(x,t) = A\sin(kx - \omega t)$ and $E_x = E_z = 0$. This is a **linearly polarized wave**.



Properties of the waves

- From $E_y = A\sin(kx - \omega t)$, we can deduce the $\vec{B}$-field using $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

- x-component: $-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$, hence $B_x = 0$

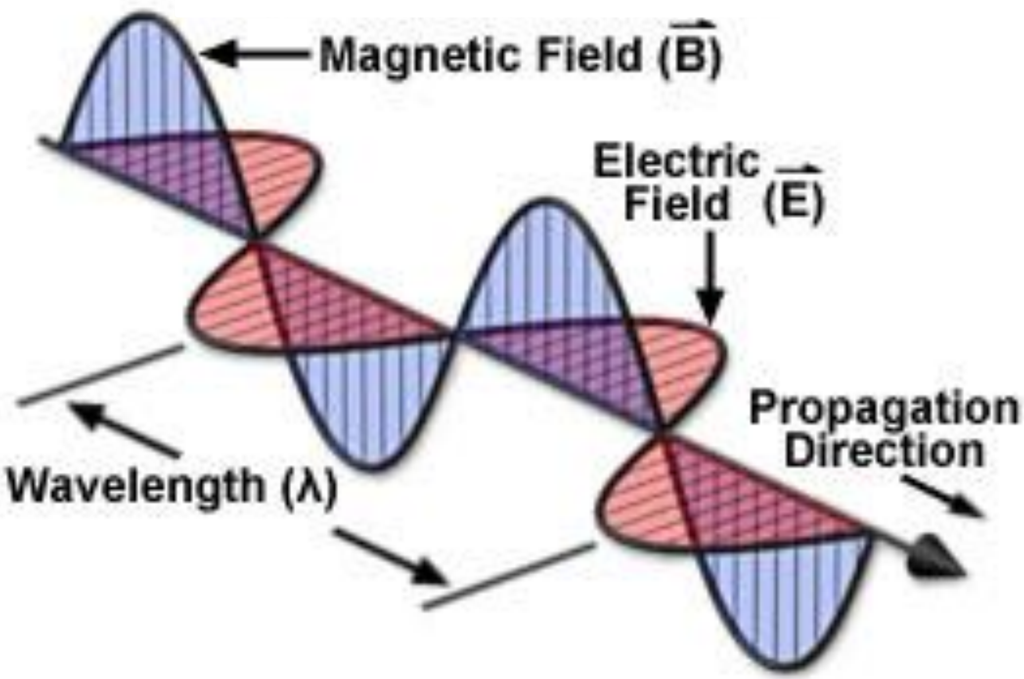
- y-component: $-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0$, hence $B_y = 0$

- z-component: $-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = kA\cos(kx - \omega t)$, hence $B_z(x,t) = \frac{A}{v}\sin(kx - \omega t) = \frac{1}{v}E_y(x,t)$

- **The magnetic field is perfectly in phase with the electric field and oriented at 90°!**

Properties of the waves

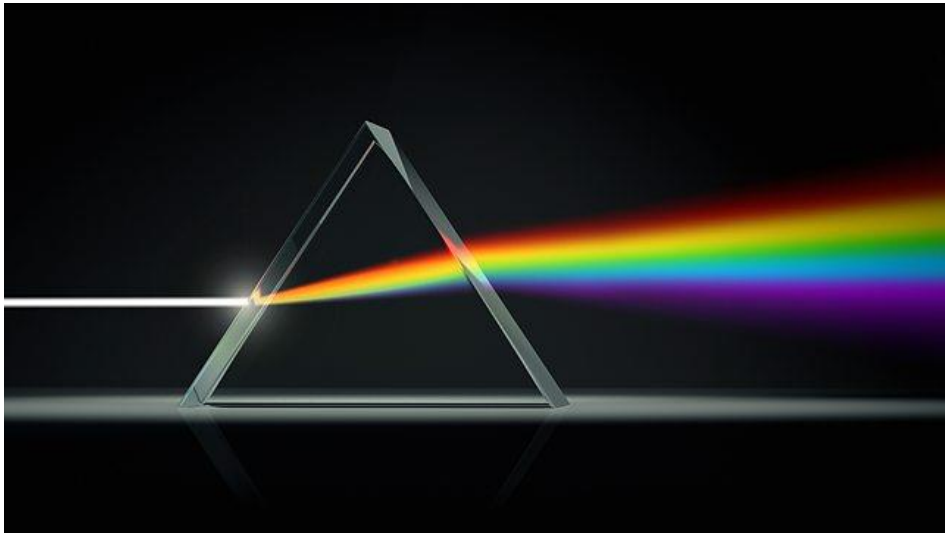
- Form of a linearly-polarized electromagnetic wave:



- *The changing $\vec{E}$-field causes a changing $\vec{B}$-field, which in its turn causes a changing $\vec{E}$-field*

Properties of the waves

- For an electromagnetic wave **travelling in a medium** with permittivity ε_r and permeability μ_r we can use Maxwell's Equations to show that **speed** $\boldsymbol{v = \frac{c}{\sqrt{\varepsilon_r \, \mu_r}}}$
- The quantity $\sqrt{\varepsilon_r \, \mu_r}$ is known as the **refractive index**

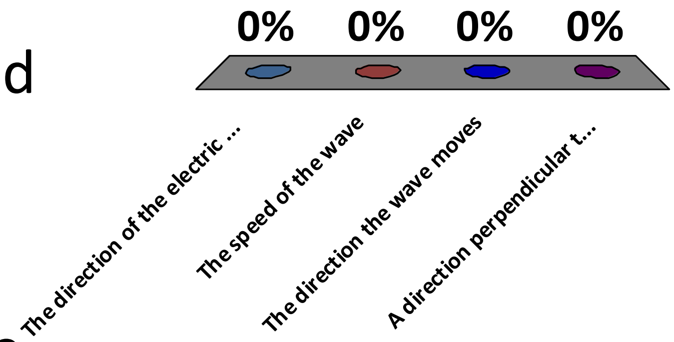


Why does this produce a spectrum of light?

Clicker question

The electric field for a plane wave is given by $\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}.\vec{r} - \omega t)$. The vector $\vec{k}$ tells you …

A. The direction of the electric field
B. The speed of the wave
C. The direction the wave moves
D. A direction perpendicular to the direction the wave moves



Clicker question

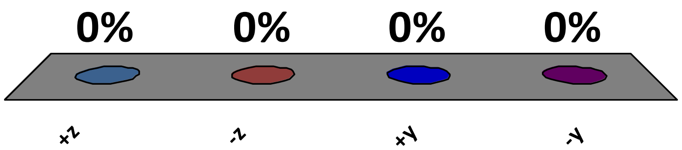
The wave propagates in the x-direction, such that $\vec{E}(x,t) = E_0\vec{\hat{y}}\cos(kx - \omega t)$. What is the direction of the magnetic field?

A. +z

B. -z

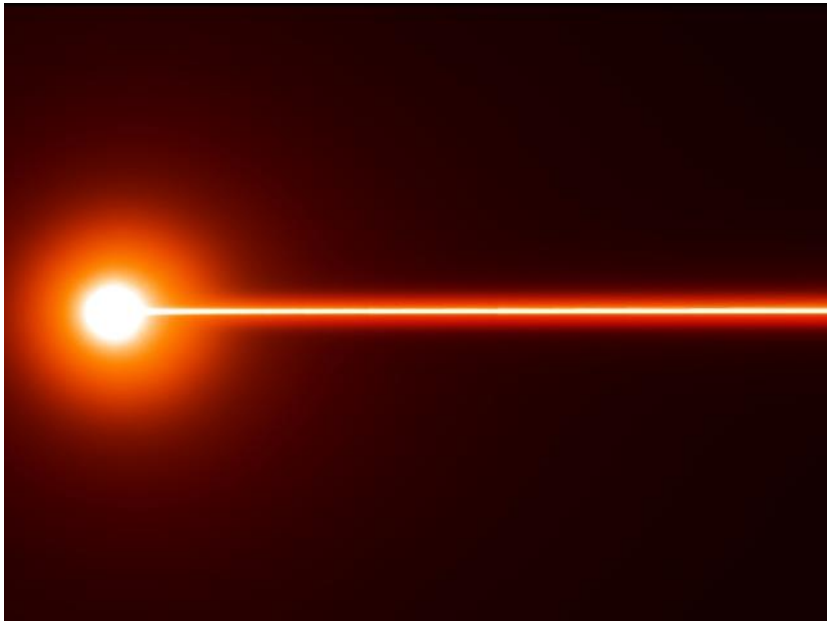
C. +y

D. -y



Energy of the waves

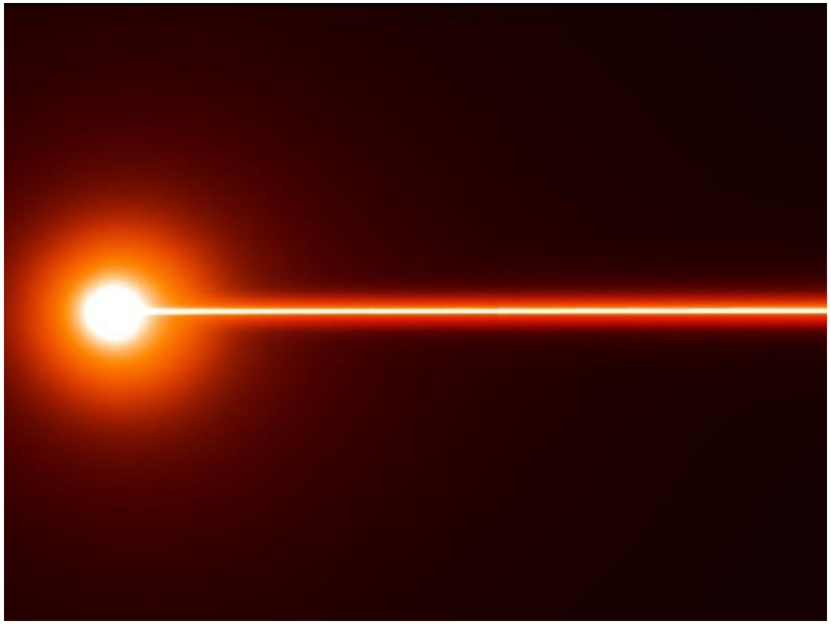
- Waves **transport energy** from one place to another



- Energy is stored in the electric field with density $\frac{1}{2}\varepsilon_0 E^2$ and the magnetic field with density $\frac{1}{2\mu_0}B^2$
- **How do these values compare?**

Energy of the waves

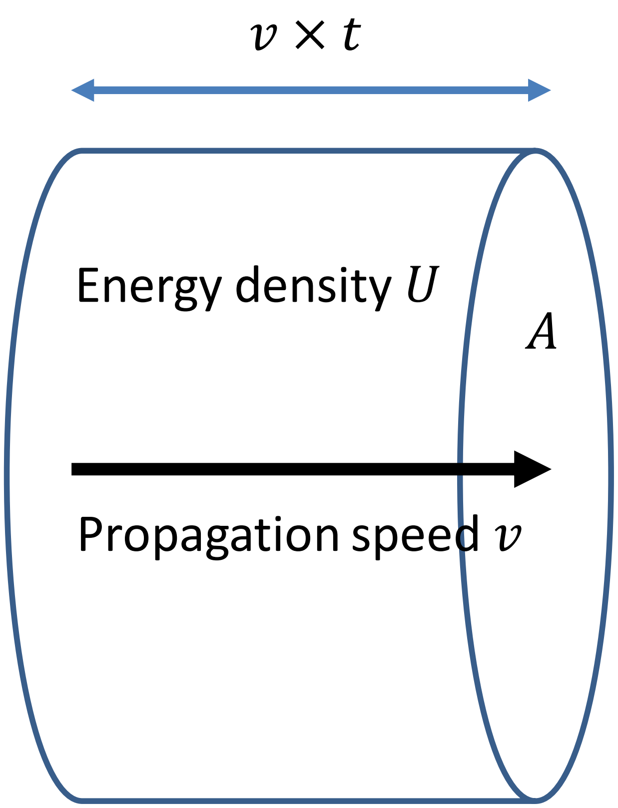
- Waves **transport energy** from one place to another



- Energy is stored in the electric field with density $\frac{1}{2}\varepsilon_0 E^2$ and the magnetic field with density $\frac{1}{2\mu_0}B^2$
- For a linearly polarized wave, $B = E/v$, where $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, hence $\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 v^2 B^2 = \frac{1}{2\mu_0}B^2$
- **The energy contained in the electric and magnetic fields is equal!**

Energy of the waves

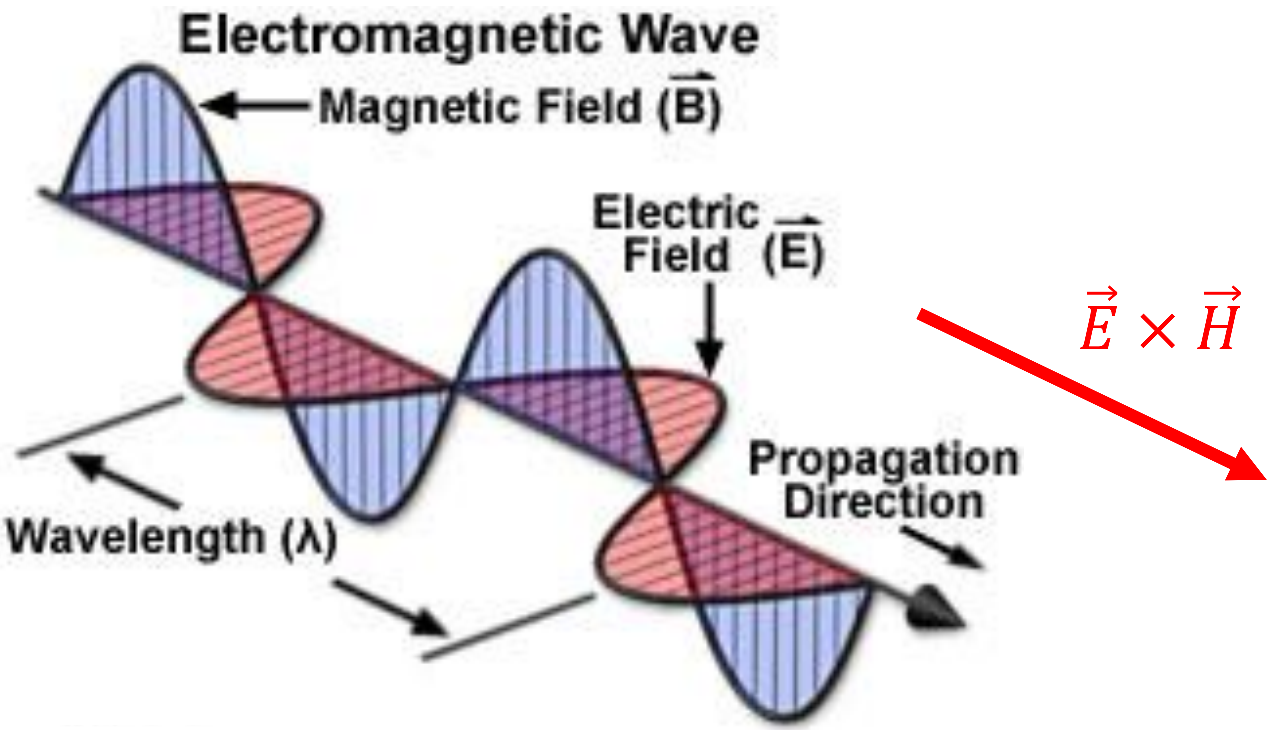
- What is the **rate of energy flow**?



- The energy flowing past area A in time t is contained in a volume Avt
- If the energy density is U, this energy is $UAvt$ hence the energy flow per unit area per unit time is equal to Uv
- For linearly-polarized electromagnetic waves, $Uv = \frac{1}{\mu_0} {B_z}^2 v = \frac{1}{\mu_0} E_y B_z$
- **The energy flow can be represented by the Poynting vector $\boldsymbol{\frac{1}{\mu_0}(\vec{E} \times \vec{B})}$**

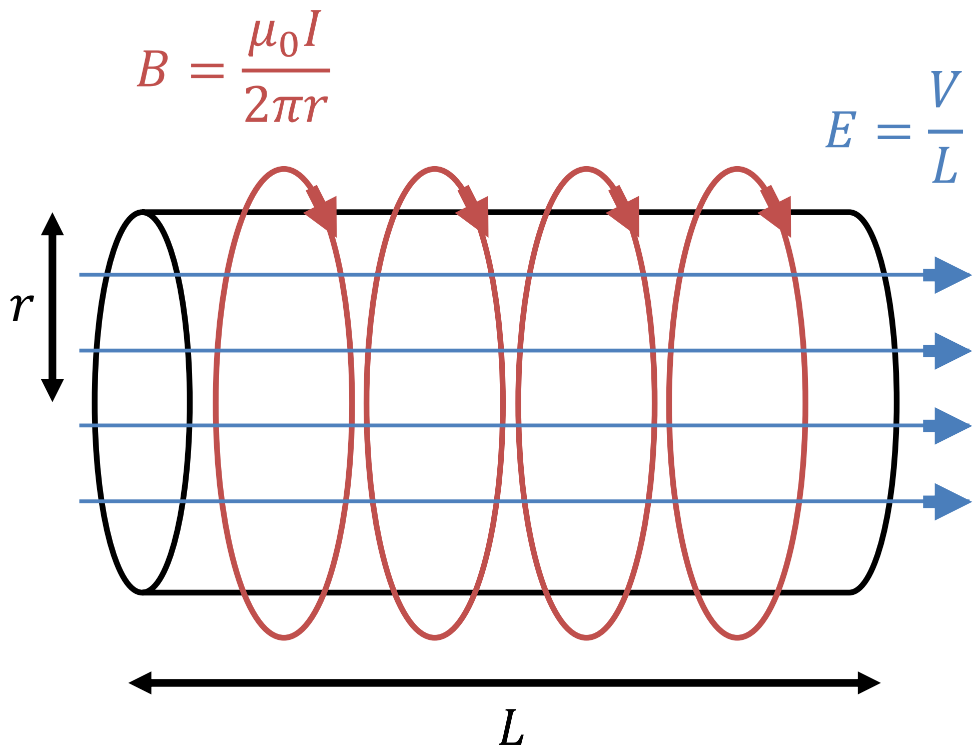
Energy of the waves

- The Poynting vector, $\frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \vec{E} \times \vec{H}$, is **oriented in the direction of travel**



Energy of the waves

- As an example, consider current I flowing along a cylindrical conductor driven by a potential V



- The energy flux is equal to the Poynting vector $\frac{1}{\mu_0}(\vec{E} \times \vec{B})$

- This is non-zero at the surface of the conductor, flowing inward

- If the curved surface area is A, the power flowing in is $P = \frac{1}{\mu_0} EBA = \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi r} 2\pi r L = VI$

- **We recover Joule heating!**

Summary

- Maxwell's equations predict that the $\vec{E}$- and $\vec{B}$-fields satisfy linked **wave equations** with speed $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
- For a linearly-polarized wave, the $\vec{E}$- and $\vec{B}$-fields are **in phase and oriented at 90°**
- The energy densities stored in the $\vec{E}$- and $\vec{B}$-fields is equal, and the energy flow per unit area per unit time is equal to the **Poynting vector** $\vec{E} \times \vec{H}$