Class 14 : Completing Maxwell's Equations

- Electromagnetism as a synthesis of ideas
- How does Faraday's Law change Maxwell's equations?
- How do the relations between fields and potentials change?
- Displacement current

Recap (1)

- **Maxwell's Equations** for the $\vec{E}$-field and $\vec{B}$-field generated by *stationary* (i.e., not varying with time) charge density ρ and current density $\vec{J}$ are:

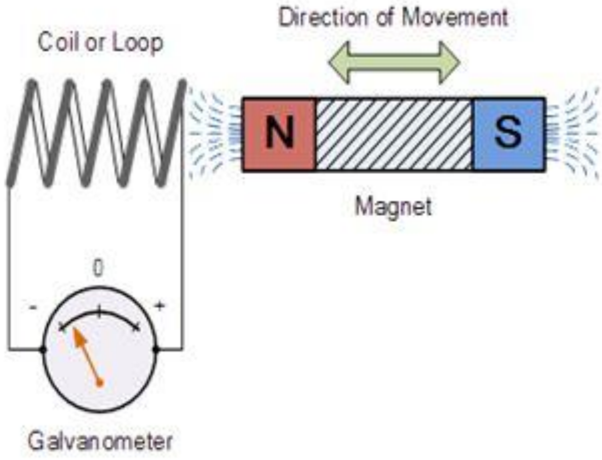
$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = \vec{0}$$

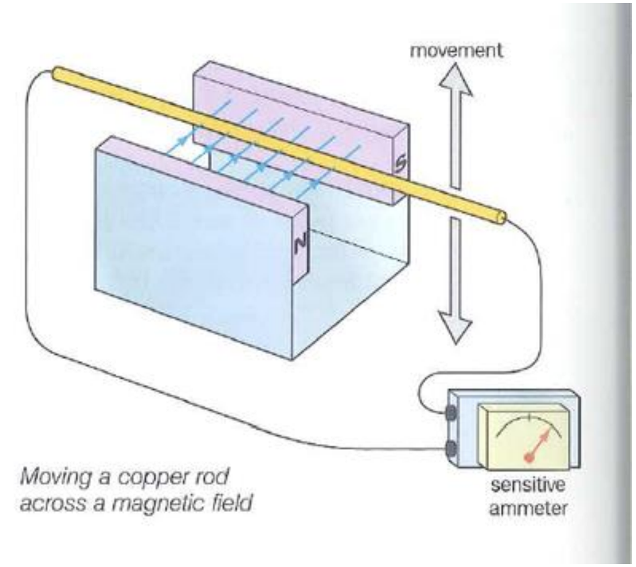
$$\vec{\nabla}.\vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Recap (2)

- Extra phenomena are produced by **time-varying** electromagnetic fields

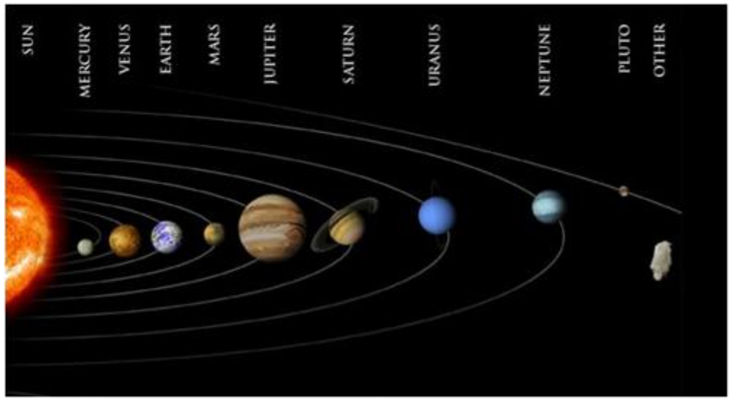
- **Faraday's Law** describes how a changing magnetic flux Φ through a circuit induces a voltage in the circuit, $V = -\frac{\partial \Phi}{\partial t}$





Synthesis of ideas

- Physics is about the **synthesis of ideas** : understanding apparently different phenomena as the joint consequences of a deeper reality

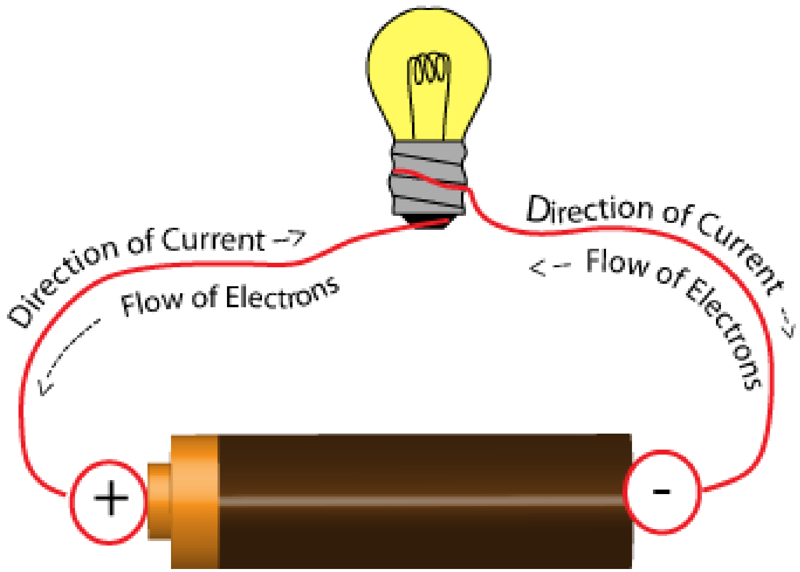


- For example, in 1666 Newton realized that the same **gravitational force** causes both apples to fall downwards from trees on the Earth, and the Earth to orbit the Sun

Synthesis of ideas

- **Electromagnetism** represents another great synthesis of ideas. Prior to 1830, *electricity* and *magnetism* were considered separate phenomena

Electricity

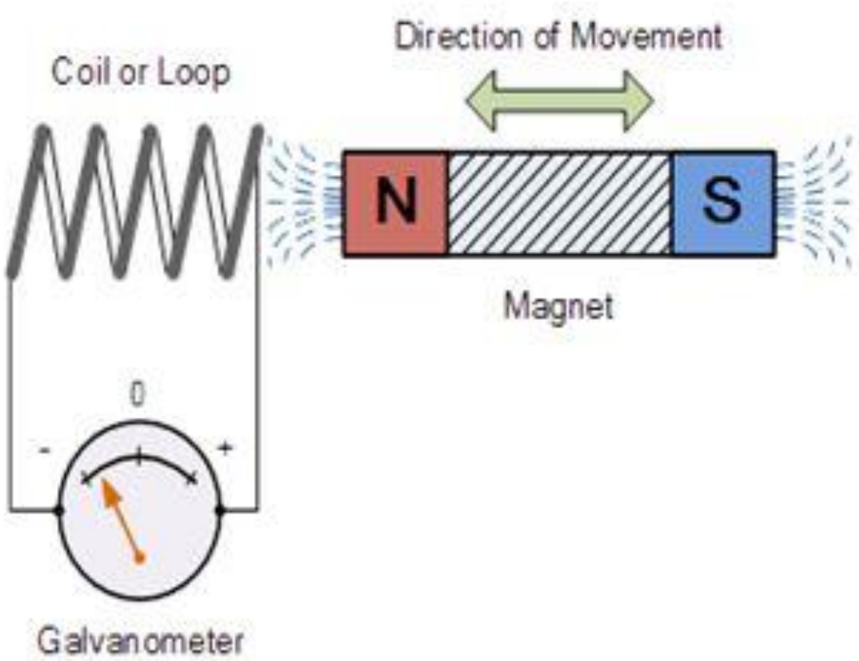


Magnetism

Synthesis of ideas

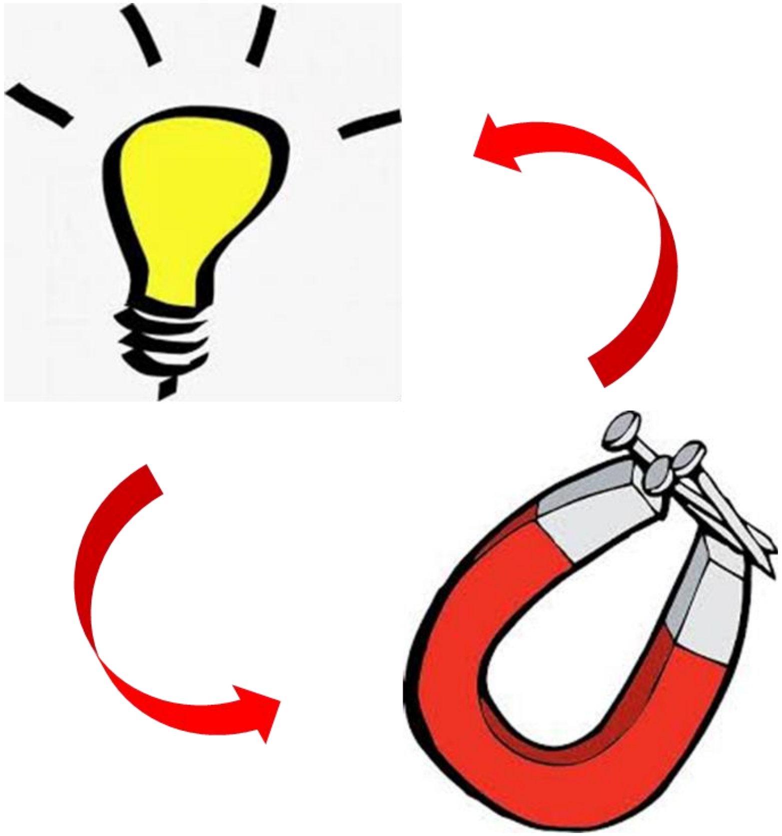
- However, Faraday's experiments demonstrated that *a magnet, as well as a battery, can drive a current*

Electromagnetism



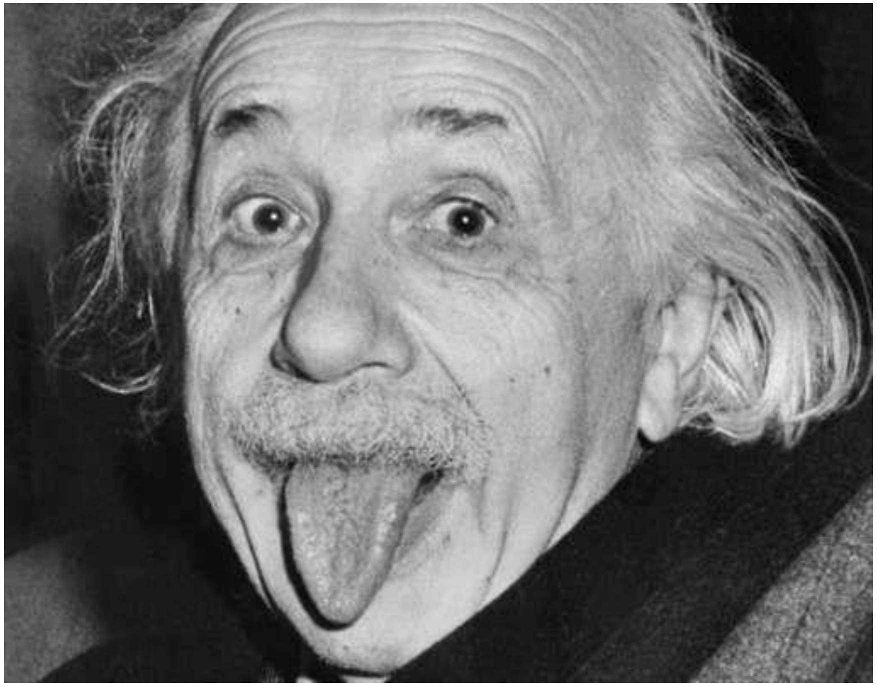
Synthesis of ideas

- Electricity and magnetism are therefore **connected by deeper principles**



Synthesis of ideas

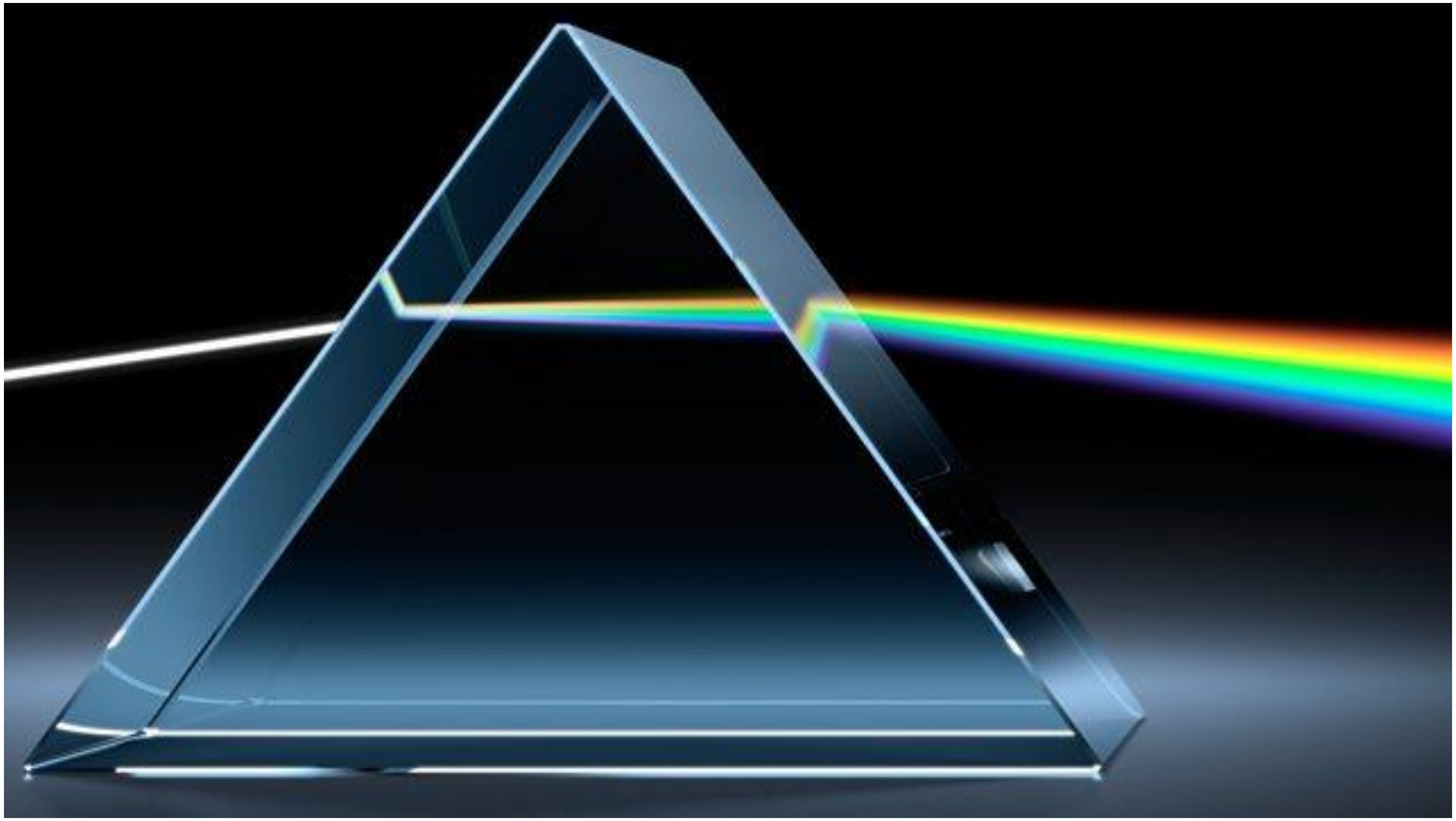
- Charges produce electric fields, and currents produce magnetic fields. But **seen in a moving reference frame, a charge becomes a current**!



- **Special relativity** – which describes how physics is viewed in different reference frames – must allow us to *transform electrostatics into magnetism*

Synthesis of ideas

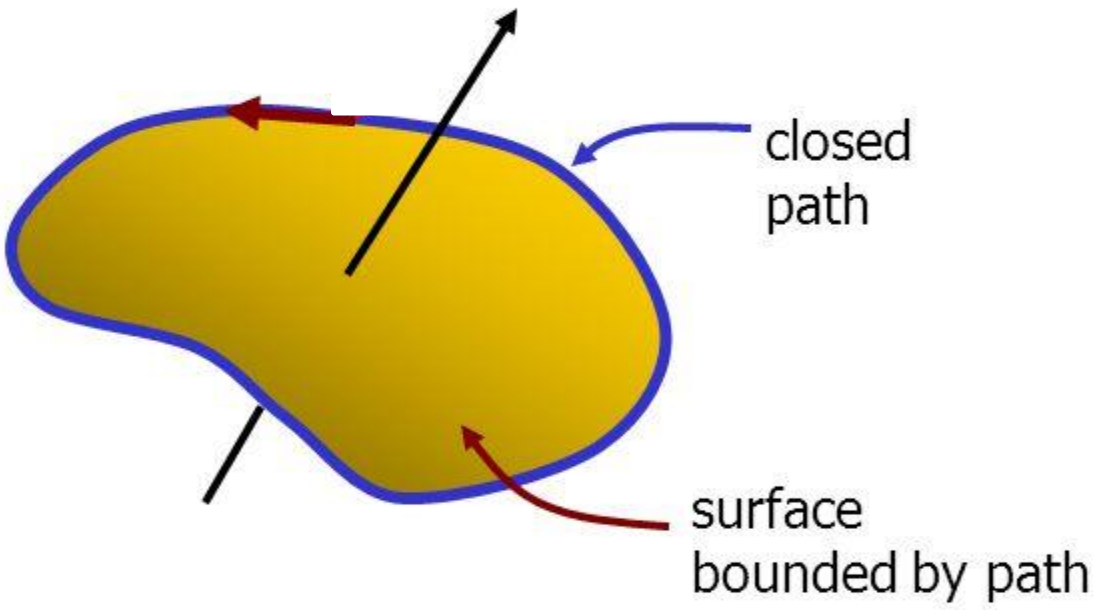
- The other great synthesis of ideas is that **electromagnetism can explain optics** (light waves)



Faraday's Law

- Faraday's Law $V = -\frac{\partial \Phi}{\partial t}$ requires a *modification to Maxwell's equation* for electrostatics, $\vec{\nabla} \times \vec{E} = \vec{0}$

- Consider a closed loop L bounding a surface S



Faraday's Law

- Faraday's Law $V = -\frac{\partial \Phi}{\partial t}$ requires a *modification to Maxwell's equation* for electrostatics, $\vec{\nabla} \times \vec{E} = \vec{0}$

- Consider a closed loop L bounding a surface S. The potential difference around the loop may be written as $V = \oint \vec{E}.d\vec{l} = \int (\vec{\nabla} \times \vec{E}).d\vec{A}$, applying Stoke's theorem

- Since $\Phi = \int \vec{B}.d\vec{A}$, Faraday's law can then be written in the form $\int \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right).d\vec{A} = 0$

- This holds true for any surface, hence $\boldsymbol{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}}$

Faraday's Law

- **What is the implication for potentials?**

- The new relation $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$ implies that we cannot derive the electric field as the gradient of an electrostatic potential, $\vec{E} = -\vec{\nabla} V$, for time-varying situations

- However, by substituting in the magnetic vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$, we can derive $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

- This is a nice way to describe how electric fields are generated by both *electrostatic charges* ($-\vec{\nabla} V$) and *changing magnetic fields* ($-\frac{\partial \vec{A}}{\partial t}$)

Displacement current

- Maxwell's Equations must also be modified in the presence of **time-varying electric fields**. The missing ingredient can be illustrated in a couple of ways.

- First, consider taking the divergence of Ampere's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. Since $\vec{\nabla} . \left(\vec{\nabla} \times \vec{B}\right) = 0$, we find $\vec{\nabla} . \vec{J} = 0$

- However, **charge conservation** that $\vec{\nabla} . \vec{J} + \frac{\partial \rho}{\partial t} = 0$

- This contradiction implies that *Ampere's law must be wrong for time-varying currents!*

Displacement current

- This is fixed by the change $\boldsymbol{\vec{\nabla}} \times \boldsymbol{\vec{B}} = \boldsymbol{\mu_0 \vec{J}} + \boldsymbol{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$

- Take the divergence of both sides. We obtain $\mu_0 \vec{\nabla}.\vec{J} + \mu_0 \varepsilon_0 \vec{\nabla}.\frac{\partial \vec{E}}{\partial t} = 0$. Cancelling μ_0 and swapping the order of $\vec{\nabla}$ and $\frac{\partial}{\partial t}$, this becomes $\vec{\nabla}.\vec{J} + \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{E}\right) = 0$

- Now substituting in $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$, we find $\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$

- *The new version of the equation is consistent with charge conservation!*

Displacement current

- For time-varying $\vec{B}$-fields : $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} + \frac{\partial \vec{\boldsymbol{B}}}{\partial \boldsymbol{t}} = \vec{\boldsymbol{0}}$

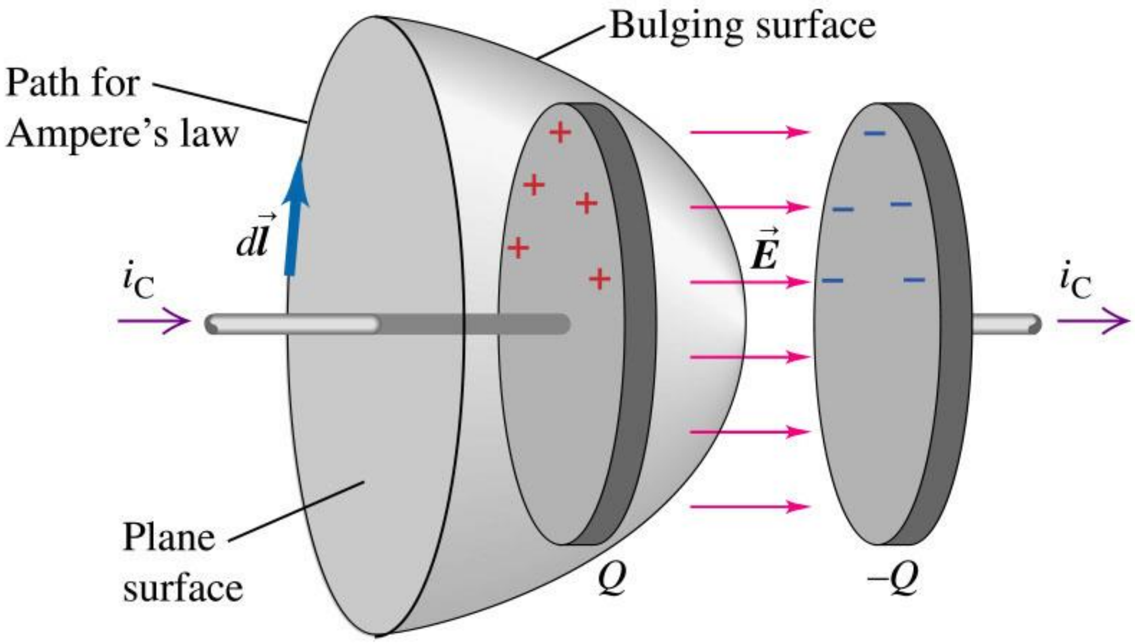
- For time-varying $\vec{E}$-fields : $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \boldsymbol{\mu_0} \vec{\boldsymbol{J}} + \boldsymbol{\mu_0 \varepsilon_0} \frac{\partial \vec{\boldsymbol{E}}}{\partial \boldsymbol{t}}$

- **A changing $\vec{B}$-field generates an $\vec{E}$-field, and a changing $\vec{E}$-field generates a $\vec{B}$-field**

- The term $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ in the 2nd equation is known as the **displacement current**, since it acts like a current density and is equal to $\frac{\partial \vec{D}}{\partial t}$, in terms of the electric displacement $\vec{D} = \varepsilon_0 \vec{E}$

Displacement current

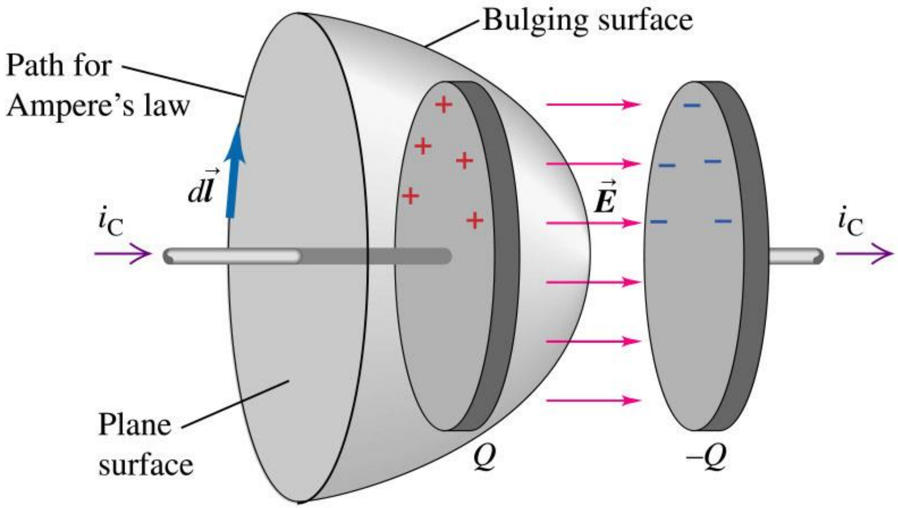
- Next, consider a **capacitor discharging into a circuit**:



- Consider applying Ampere's Law $\oint \vec{B}.d\vec{l} = \mu_0 I_{enclosed}$ to both the *plane surface* and the *bulging surface* in the diagram

Displacement current

- The current enclosed by these two surfaces is different – we again have a problem!



- The situation can be reconciled by including the **displacement current** enclosed by the bulging surface
- The electric field $E = \sigma/\varepsilon_0$ in terms of the charge density σ
- If A is the plate area, then $Q = \sigma A$ hence $I = \frac{dQ}{dt} = \varepsilon_0 \frac{dE}{dt} A$
- This is equal to the displacement current enclosed by the surface!

Maxwell's complete equations

- Maxwell's Equations are now complete!

- In a vacuum:

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

Maxwell's complete equations

- More generally, in materials with permittivity ε_r and permeability μ_r we can use the electric displacement $\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$ and magnetic intensity $\vec{H} = \vec{B}/\mu_r \mu_0$:

$$\vec{\nabla}.\vec{D} = \rho \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

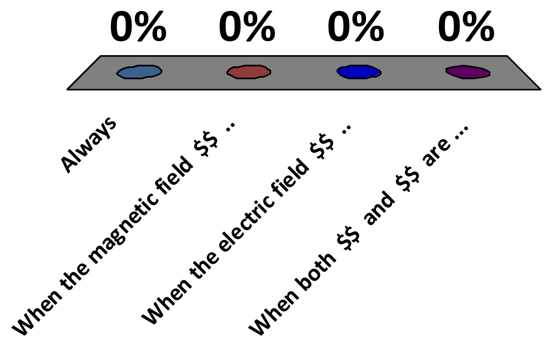
Clicker question

Ampere's Law

$$\oint \vec{B}.d\vec{l} = \mu_0 I_{enclosed}$$

holds true …

A. Always
B. When the magnetic field $\vec{B}$ is not changing
C. When the electric field $\vec{E}$ is not changing
D. When both $\vec{E}$ and $\vec{B}$ are not changing



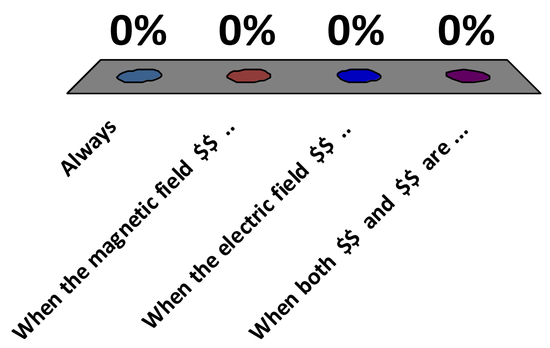
Clicker question

Gauss's Law

$$\int \vec{E}.\,d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

holds true …

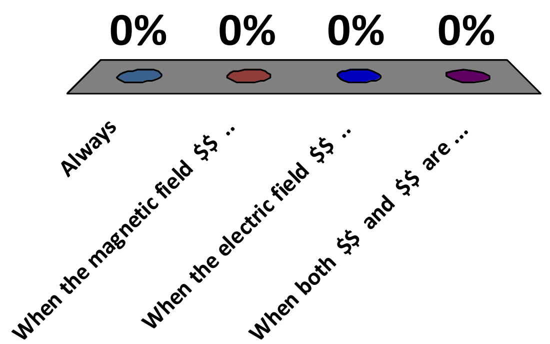
A. Always
B. When the magnetic field $\vec{B}$ is not changing
C. When the electric field $\vec{E}$ is not changing
D. When both $\vec{E}$ and $\vec{B}$ are not changing



Clicker question

An electrostatic potential $V = -\int_{\infty}^{P} \vec{E}.d\vec{l}$ can be defined …

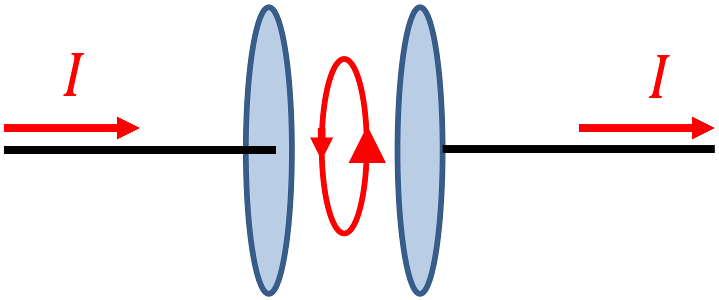
A. Always
B. When the magnetic field $\vec{B}$ is not changing
C. When the electric field $\vec{E}$ is not changing
D. When both $\vec{E}$ and $\vec{B}$ are not changing



Clicker question

A capacitor is being charged.

What is the value of $\oint \vec{B}.d\vec{l}$ around the loop shown?

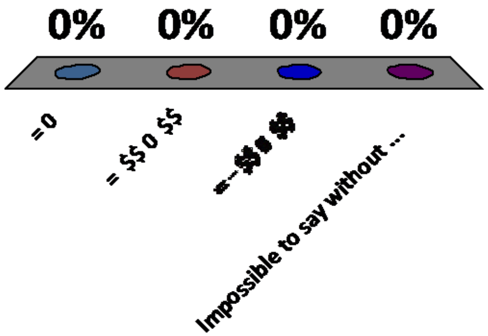


A. $= 0$

B. $= \mu_0 \, I$

C. $= -\mu_0 \, I$

D. Impossible to say without knowing surface



Summary

- An electric field is generated by a changing magnetic field (**electromagnetic induction**)
- A magnetic field is also generated by a changing electric field, as described by the **displacement current** such that $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$