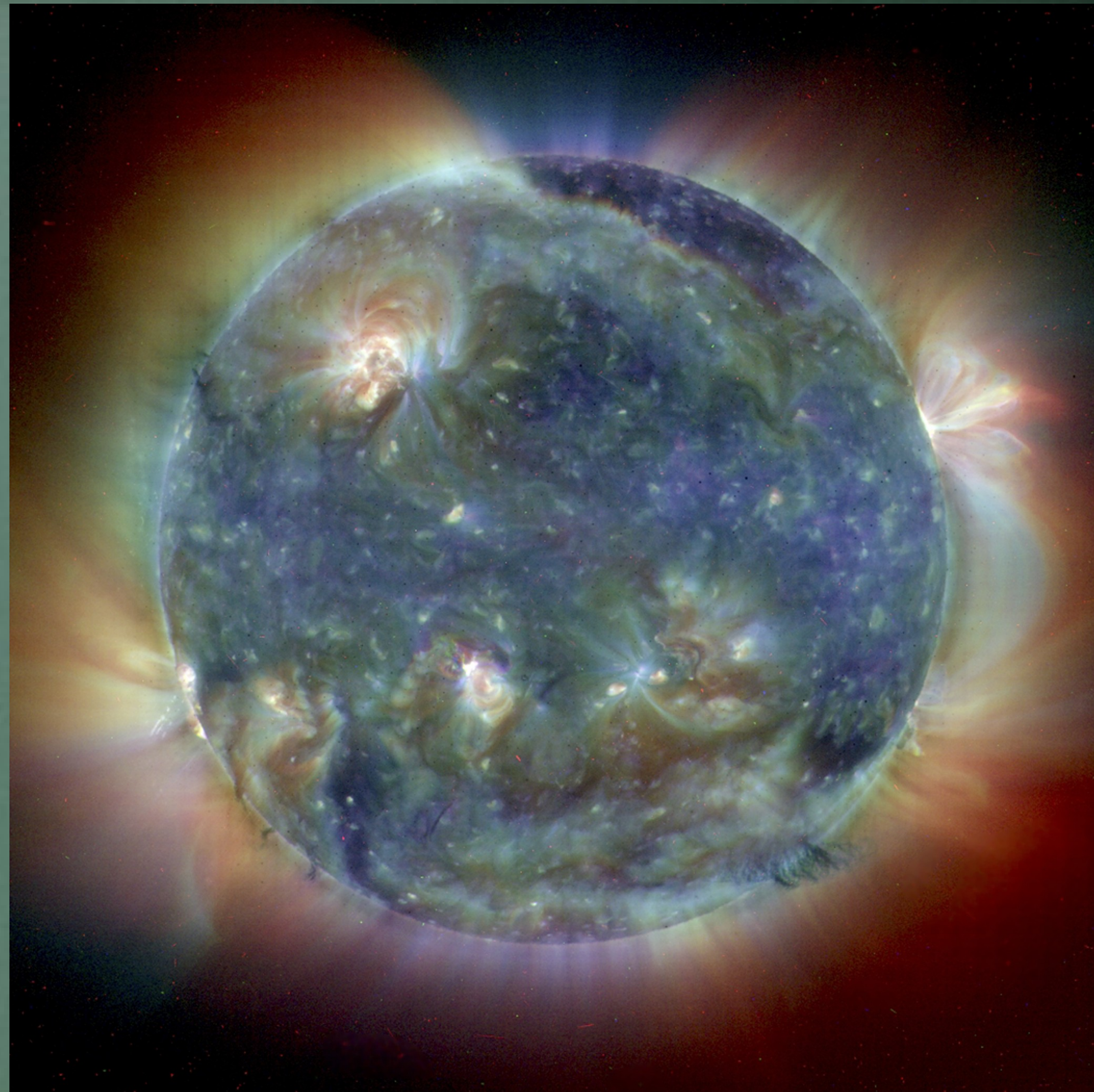
Magnetohydrodynamics Basics

or "MHD in one lecture?! You've gotta be kidding!"

Paul Cally, Centre for Stellar & Planetary Astrophysics, Monash University

- MHD Equations
- Physical Interpretation
- MHD Waves

Solar atmosphere dominated by magnetic fields



MHD

- MHD = fluid dynamics + Maxwell's eqns - displacement current + Ohm's "Law" [$j=\sigma(E+v \times B)$ or generalization]
- Applies to (at least partially) ionized gases (plasmas)
- Nonrelativistic
- Assumes highly collisional, low frequency (c.f. cyclotron frequency)

MHD Approximation

- $v^2 \ll c^2$
- Then Ampère's Law simplifies:

$$\nabla \times B = \mu j + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

The electric field E plays no independent role in MHD: it is entirely derivable from v and B:

$$E = -v \times B + \frac{j}{\sigma} = -v \times B + \frac{\nabla \times B}{\mu \sigma}$$

MHD Equations

- Continuity $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

- Momentum $$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{j} \times \boldsymbol{B} + \text{viscous terms}$$

Lorentz force

where $j = \mu^{-1} \nabla \times B$ = current density; $\mu = 4\pi \times 10^{-7}$ (SI units)

- Energy (many different forms)

$$\rho T \frac{Ds}{Dt} = \text{viscous heating + Joule heating - radiative losses - conductive losses} + \ldots$$

- Induction (σ=electrical conductivity; η=magnetic diffusivity)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \quad \text{with} \quad \nabla \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \eta = 1/\mu\sigma$$

Magnetic Reynolds Number

- Induction eqn: $$\frac{\partial \boldsymbol{B}}{\partial t} = \underbrace{\nabla \times (\boldsymbol{v} \times \boldsymbol{B})}_{advective} + \underbrace{\eta \nabla^2 \boldsymbol{B}}_{diffusive}$$
- Ratio of advective to diffusive terms:

$$R_m = \frac{V\,L}{\eta}$$

  - the Magnetic Reynolds Number
  - V=typical velocity, L=typical lengthscale
  - $R_m >> 1 \Rightarrow$ dominated by advection
  - $R_m << 1 \Rightarrow$ dominated by diffusion

Ideal MHD Equations

- Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

- Momentum

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{j} \times \boldsymbol{B} + \cancel{\text{viscous terms}}$$

- Energy (many different forms)

$$\rho T \frac{Ds}{Dt} = \cancel{\text{viscous heating + Joule heating - radiative losses - conductive losses + …}}$$

$$i.e., \quad \frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \qquad (\text{adiabatic};\, c = \sqrt{\gamma\, p/\rho} = \text{sound speed})$$

- Induction

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \cancel{\eta \nabla^2 \boldsymbol{B}} \qquad \text{with} \qquad \nabla \cdot \boldsymbol{B} = 0$$

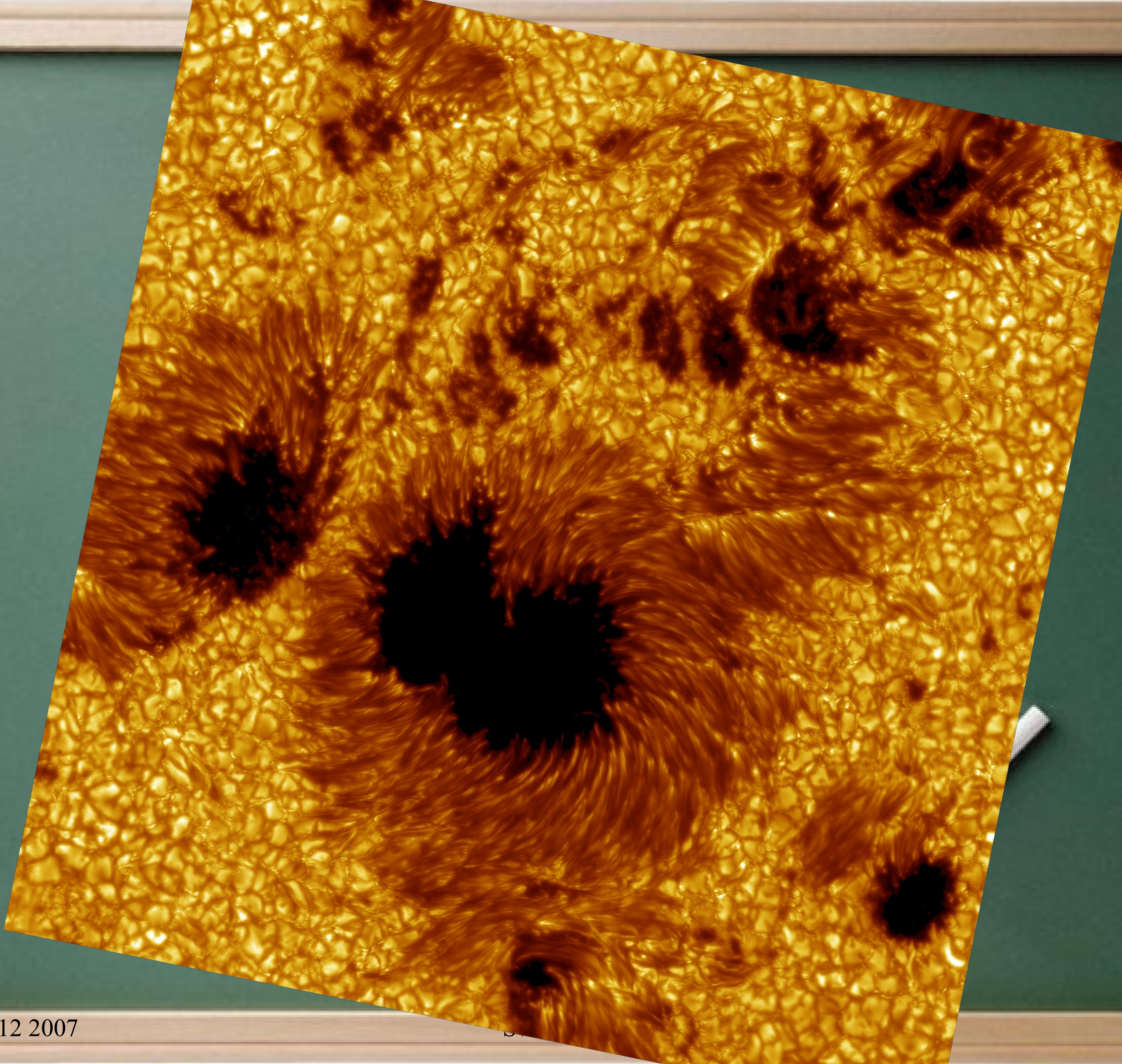
Alfvén's Theorem: flux freezing

In <u>Ideal</u> MHD, the magnetic field and the fluid are frozen together.

That is: two fluid elements originally connected by a field line are always so connected.

So what?

- If the field is very strong ($\beta<<1$), plasma is constrained to flow along it;
- If the field is weak ($\beta>>1$), it is advected with the plasma.

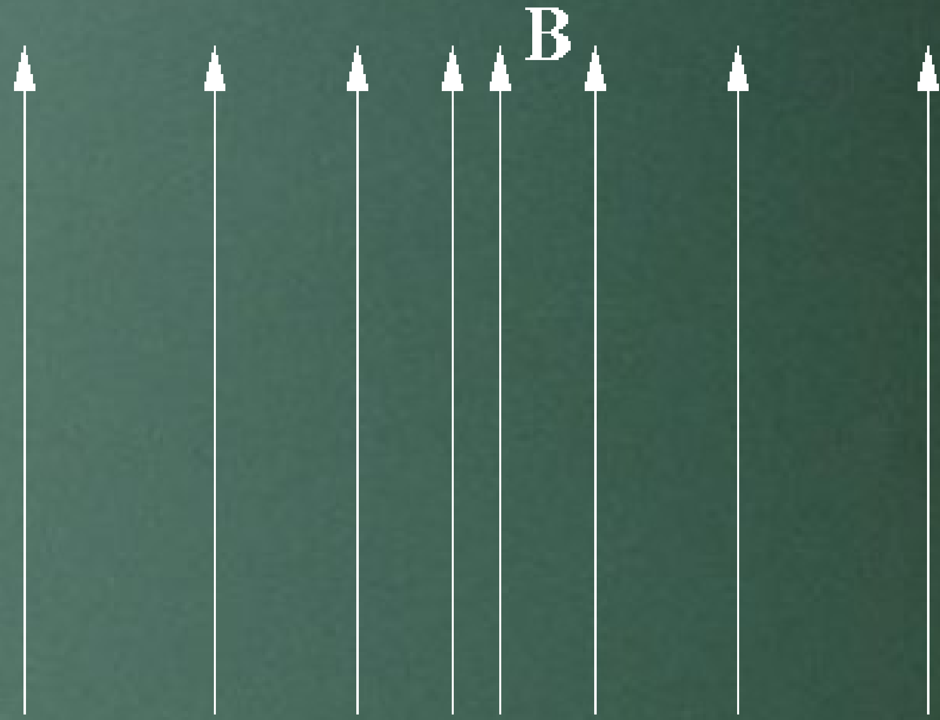


Lorentz Force jXB

$$
\begin{aligned}
\boldsymbol{j}\times\boldsymbol{B} &= \frac{1}{\mu}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \\
&= \frac{1}{\mu}\boldsymbol{B}\cdot\nabla\,\boldsymbol{B} - \nabla\frac{B^2}{2\mu} \\
&= \nabla\cdot\underbrace{\left(\frac{\boldsymbol{B}\,\boldsymbol{B}}{\mu} - \frac{B^2}{2\mu}\boldsymbol{I}\right)}_{\text{magnetic stress tensor}}
\end{aligned}
$$

- Stress:
  - tension along field lines B^2/μ
  - isotropic pressure $p_{mag} = B^2/2\mu$
- $$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla(p + p_{mag}) + \rho\,\boldsymbol{g} + \frac{1}{\mu}\boldsymbol{B}\cdot\nabla\,\boldsymbol{B} + \text{~~viscous terms~~}$$

Example 1



- What do you expect this to do?

$$\mathbf{B} = (0, e^{-x^2}, 0)$$
$$\Rightarrow \quad \mathbf{j} = (0,0,2x\,e^{-x^2})/\mu$$
$$\Rightarrow \quad \mathbf{j}\times\mathbf{B} = (2x\,e^{-2x^2}, 0,0)/\mu$$

… or

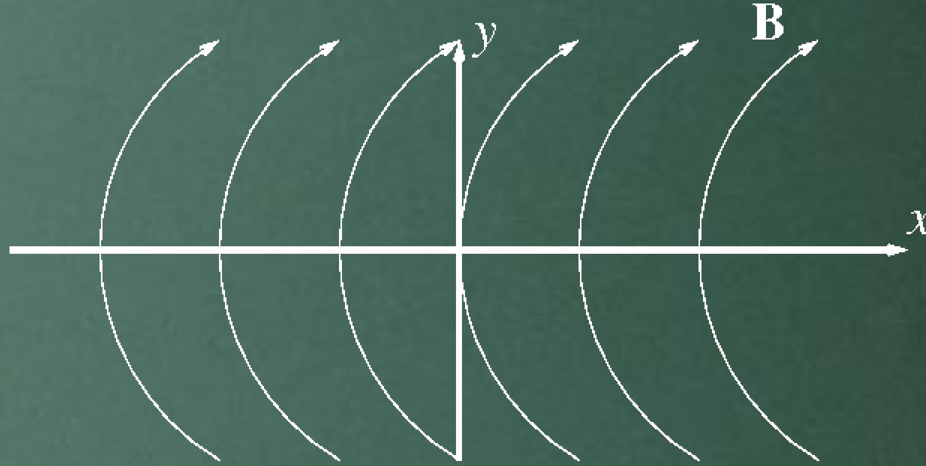
$$\mathbf{B}\cdot\nabla\,\mathbf{B} = \mathbf{0} \quad \text{and} \quad \nabla B^2/2\mu = (-2x\,e^{-2x^2}, 0,0)/\mu$$
$$\Rightarrow \quad \mathbf{j}\times\mathbf{B} = \mathbf{B}\cdot\nabla\,\mathbf{B}/\mu - \nabla B^2/2\mu = (2x\,e^{-2x^2}, 0,0)/\mu$$

- Pressure pushing away from high B.

Example 2

- And this?

$$\boldsymbol{B} = (2y, 1, 0)$$
$$\Rightarrow \boldsymbol{j} = (0, 0, -2)/\mu$$
$$\Rightarrow \boldsymbol{j} \times \boldsymbol{B} = (2, -4y, 0)/\mu$$



- … or

$$\boldsymbol{B} \cdot \nabla \boldsymbol{B} = (2, 0, 0)$$
$$\nabla B^2 = (0, 8y, 0)$$

So $$\frac{1}{\mu} \boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla \frac{B^2}{2\mu} = \left(\frac{2}{\mu}, 0, 0\right) - \left(0, \frac{4y}{\mu}, 0\right)$$

- "Slingshot effect" to the right! – important in flares and CMEs

Plasma Beta

- A simple measure of the relative importance of the gas and magnetic forces $\beta = \frac{p_{gas}}{p_{mag}}$
- If β >> 1 the gas dominates (e.g. stellar interiors)
- If β << 1 the magnetic field dominates (e.g. stellar coronae)
- If β ~ 1 then get complicated interaction of field & fluid (e.g. sunspots)

MHD Waves

- All waves result from a restoring force
- Sound waves are produced by compression (gas pressure is restoring force)
- Magneto-acoustic waves derive from gas & magnetic pressure, and magnetic tension
- Alfvén waves produced by magnetic tension only.

MHD Waves: Linear Theory

Assume uniform medium, no gravity, ...

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{v} = 0$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p_1 + \frac{1}{\mu} (\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0$$

$$\frac{\partial p_1}{\partial t} = c^2 \frac{\partial \rho_1}{\partial t}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0)$$

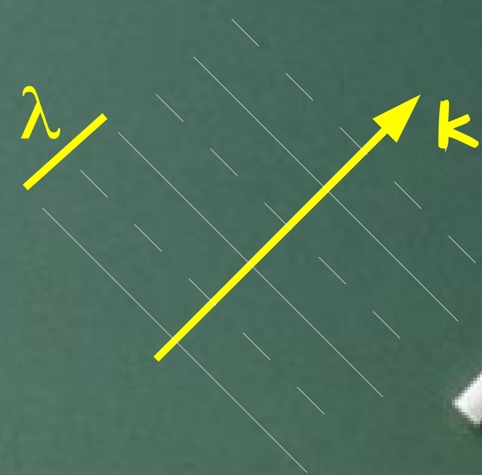
Set v=V exp[i(k·r-ωt)] and simila

etc, so $\nabla = i\boldsymbol{k}$ and $\partial/\partial t = -i\omega$, to get

$$\omega^2 \boldsymbol{V} = c^2 \boldsymbol{k} (\boldsymbol{k} \cdot \boldsymbol{V}) + \frac{1}{\mu \rho_0} (\nabla \times (\nabla \times (\boldsymbol{V} \times \boldsymbol{B}_0))) \times \boldsymbol{B}_0$$

Basic Waves: Revision

- Circular Frequency ω (radians per second)
- Frequency $\nu=\omega/2\pi$ measured in Hz.
- Period (seconds) = $1/\nu=2\pi/\omega$
- Wave vector **k** specifies the direction ⊥ to wavefronts
- Wavelength $\lambda=2\pi/|\mathbf{k}|$



Careful …

Three velocities, all (generally) different:

- Plasma velocity v
  - motion of the plasma itself
- Phase velocity $v_{ph} = \frac{\omega}{k}\hat{\boldsymbol{k}}$
  - velocity of the wave pattern
- Group velocity $v_{gr} = \frac{\partial \omega}{\partial \boldsymbol{k}}$
  - velocity of energy propagation

Sound Wave

- Set $B_0 = 0$ $\omega^2 \boldsymbol{V} = c^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{V})$
- Dot both sides with **k** to get dispersion relation $\omega^2 = c^2 k^2$
- Phase speed $\frac{\omega}{k} = \pm c$ ("pattern" speed)
- Group velocity $\frac{\partial \omega}{\partial \boldsymbol{k}} = c\hat{\boldsymbol{k}}$ (energy propagation speed & direction)
- Note: **V** ‖ **k**, i.e., wave <u>longitudinal</u>

Alfvén Wave

- Driven purely by magnetic tension
- Incompressive, i.e. $\nabla \cdot v = 0$, i.e. $k \cdot v = 0$, i.e. <u>transverse</u> (like guitar string)
- Then $\omega^2 \mathbf{V} = \cancel{c^2 \mathbf{k}(\mathbf{k}\cdot\mathbf{V})} + \frac{1}{\mu \rho_0}(\nabla \times (\nabla \times (\mathbf{V} \times \mathbf{B}_0))) \times \mathbf{B}_0$

$$\text{i.e.} \quad \omega^2 V^2 = \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu \rho_0} V^2$$

$$\text{i.e.} \quad \omega^2 = a^2 k_{\|}^2$$

where $a^2 = B^2/\mu\rho_0$ defines the Alfvén speed a

- Note: $B_0 \cdot v = 0$, i.e. $v \perp$ to <u>both</u> B_0 and k

Alfvén Wave

- Phase speed $\frac{\partial \omega}{\partial k} = \pm a \cos\theta$ where θ is the angle between **B** and **k**
  - v_{ph}=a for parallel propagation, 0 for perpendicular!
- Group velocity $\frac{\partial \omega}{\partial \boldsymbol{k}} = \pm a \hat{\boldsymbol{B}}_0$
  - i.e. energy travels <u>directly along fieldlines</u> at speed a
- Obvious! It's a tension driven wave.

Magnetoacoustic Waves

- Both pressures and magnetic tension

$$\omega^2 \boldsymbol{V} = c^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{V}) + \frac{1}{\mu\rho_0}(\nabla\times(\nabla\times(\boldsymbol{V}\times\boldsymbol{B}_0)))\times\boldsymbol{B}_0$$

$$= c^2 \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{V}) + \frac{1}{\mu\rho_0}\left[(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2\boldsymbol{V} - (\boldsymbol{k}\cdot\boldsymbol{B}_0)(\boldsymbol{k}\cdot\boldsymbol{V})\boldsymbol{B}_0 - (\boldsymbol{k}\cdot\boldsymbol{B}_0)(\boldsymbol{V}\cdot\boldsymbol{B}_0)\boldsymbol{k} + B_0^2(\boldsymbol{k}\cdot\boldsymbol{V})\boldsymbol{k}\right]$$

- Dot with k and with B_0

$$\left[\omega^2 - (a^2+c^2)k^2\right]\boldsymbol{k}\cdot\boldsymbol{V} + a^2 k^2 k_\parallel \boldsymbol{V}\cdot\hat{\boldsymbol{B}}_0 = 0$$

$$c^2 k_\parallel \boldsymbol{k}\cdot\boldsymbol{V} - \omega^2 \boldsymbol{V}\cdot\hat{\boldsymbol{B}}_0 = 0$$

- Eliminate k·V to get magnetoacoustic dispersion relation

$$\omega^4 - (a^2+c^2)k^2\omega^2 + a^2c^2k^2k_\parallel^2 = 0$$

Magnetoacoustic Waves

- Solve Biquadratic:

fast

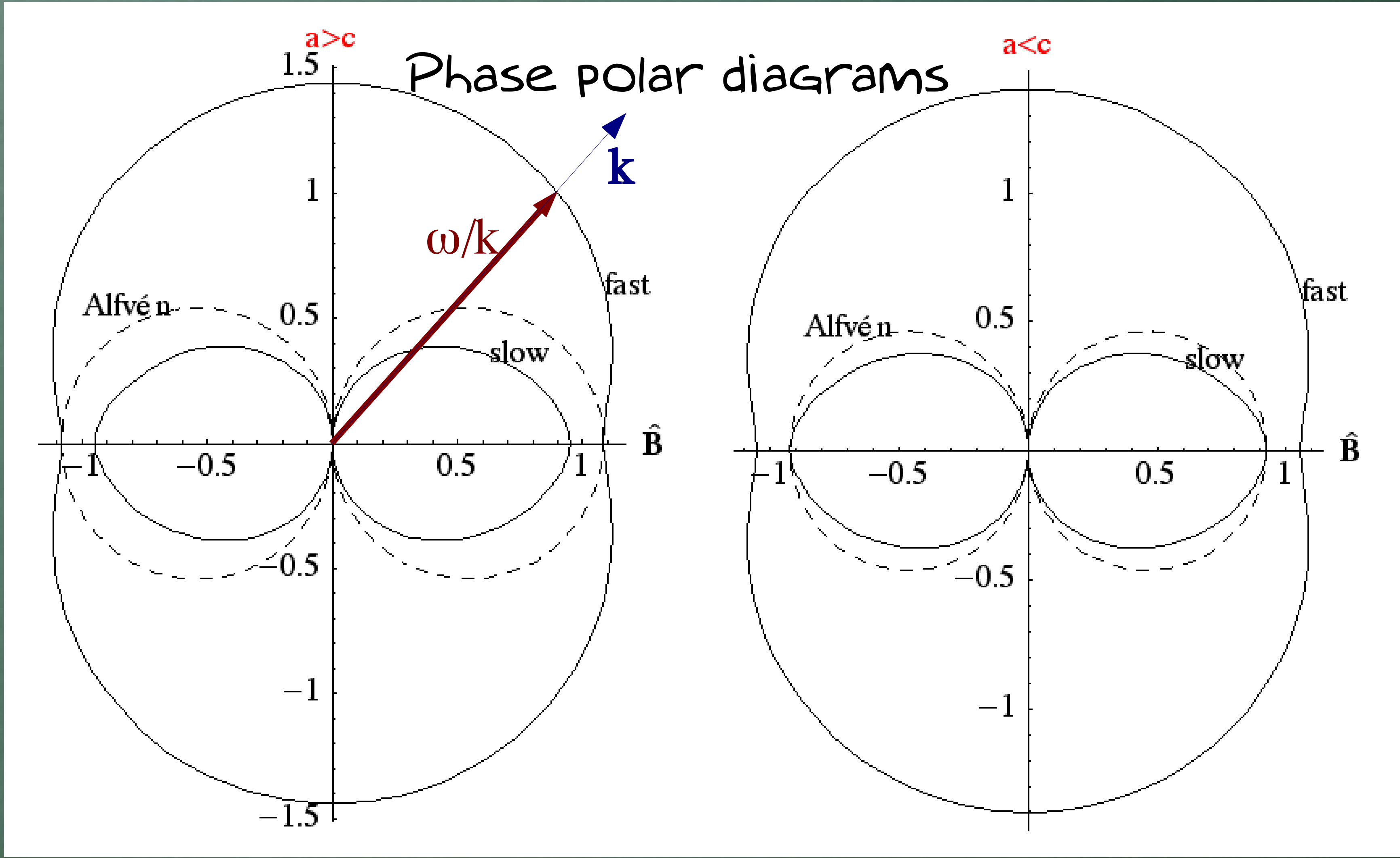
$$\frac{\omega}{k} = \left[\frac{1}{2}(a^2+c^2) \pm \sqrt{(a^2+c^2)^2 - 4a^2c^2\cos^2\theta}\right]^{1/2}$$

slow

- Two types of wave: fast & slow
  - NB: they involve both a & c

Fast & Slow: Phase Speed

- Phase speed depends on direction of **k**

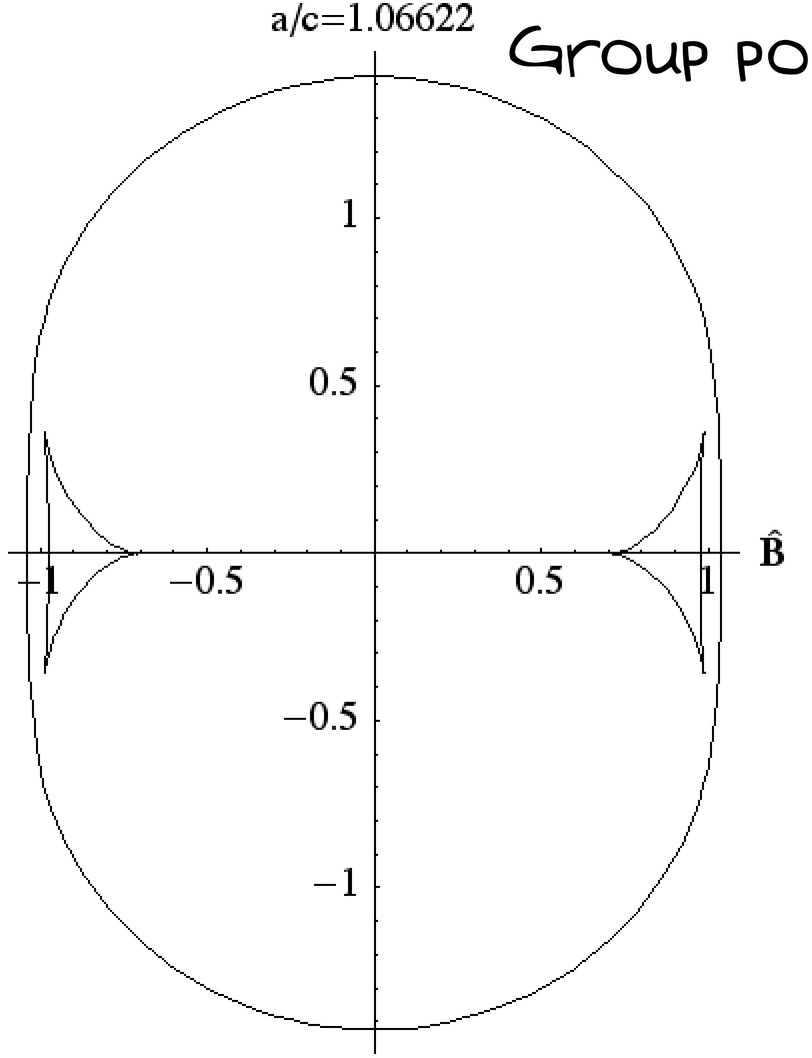


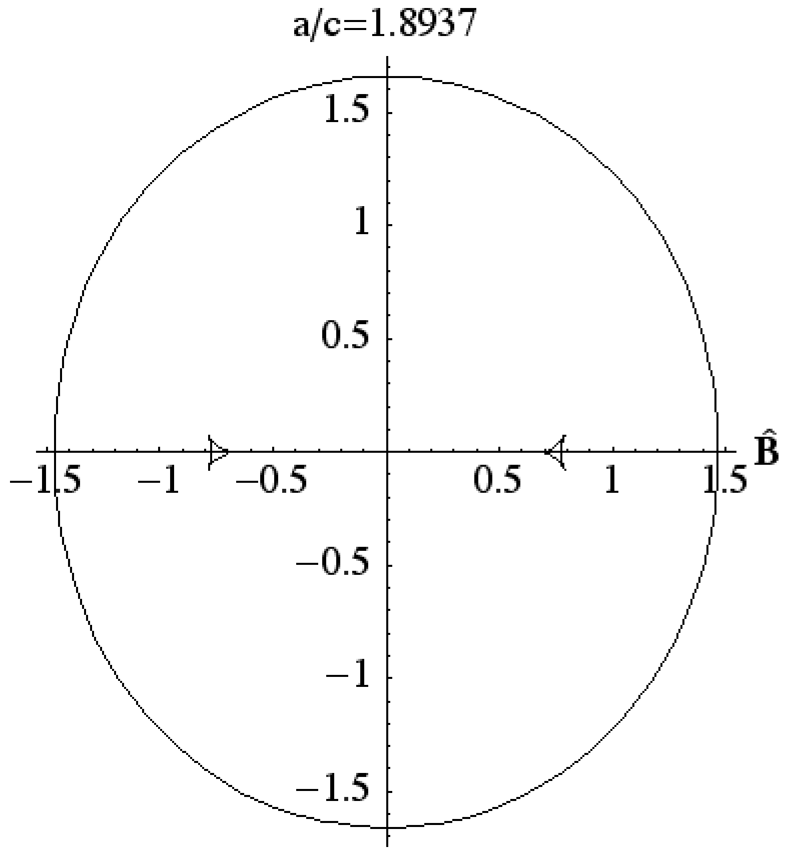
Fast & Slow: Phase Speed

- Phase speed depends on direction of **k**
- Fast wave is fastest perpendicular to field and slowest along it (why?)
  - in range $(a^2+c^2)^{1/2} \geq \omega/k \geq \max(a,c)$
- Slow wave is fastest along field lines
  - in range $\min(a,c) \geq \omega/k \geq 0$
- The Alfvén wave is "intermediate".

Fast & Slow: Group Speed

Group polar diagrams





Fast & Slow: Group Speed

- Energy travels fastest across field lines for fast wave
- Slow wave energy transport along field lines restricted to a cone about B
  - cone gets tighter as a/c departs further from 1.

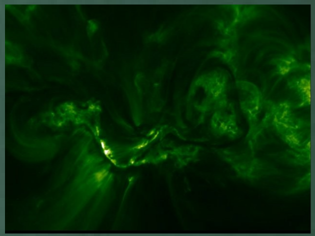
Q: How Many Wave Types?

A: Three — Alfvén, fast, & slow (6th order dispersion relation).

- But what about the sound wave?
  - No! The sound wave is just a magnetoacoustic wave (fast or slow depending if $c>a$ or $a>c$) travelling directly along the field lines. Because it is longitudinal, it does not bend or displace them, so there is no magnetic effect.

Non-Ideal Effects

- Magnetic reconnection
  - Solar corona (flares, CMEs)
    - massive source of energy
  - Earth's magnetosphere (interaction with solar wind)
- … another day perhaps.



Lunchtime!